Local Morse Theory for Solutions to the Heat Equation and Gaussian Blurring

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Received February 5, 1992

A general problem in partial differential equations is to describe the generic behavior of solutions to a partial differential equation, e.g., \( P(u) = g \), where \( P \) is a differential operator (usually with some boundary conditions). Often the results that are known result from maximum principles, various estimates, etc. Very little is known about the generic local behavior of such solutions. In this paper we apply methods from singularity theory to answer this question for the heat equation (these results were originally announced in [D1]) and in the process indicate how the methods apply more generally to a large collection of partial differential equations (even allowing nonconstant coefficients), see [D4].

We are interested in a problem that arises in computer vision. Here one wishes to be able to identify and manipulate the significant objects in a computer screen image. A particular method of doing this is to use "Gaussian blurring" applied to a "pixel intensity" function \( u_0(x) \) which describes the brightness of the various pixels on the screen (see Witkin [W], Koenderink [K], and more recently Lifshitz [L], Pizer et al. [PGL]). Its aim is to blur away nonimportant details of the image at various levels of scale. When Gaussian blurring is applied to an intensity function \( u_0(x) \), it yields a family \( u(x, t) \) of intensity functions parametrized by \( t \). The family \( u(x, t) \) is actually a solution to the heat equation

\[
\frac{\partial u}{\partial t} = \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \quad \text{on an open} \quad U \subset \mathbb{R}^n \times \mathbb{R}_+
\]

(1)

and satisfying initial conditions \( u(x, 0) = u_0(x) \) for \( u_0: \mathbb{R}^n \to \mathbb{R} \) at least continuous.

We would like to know what the generic properties of \( u \) are, i.e., for most initial functions \( u_0 \), what local properties we can expect \( u \) to have. This would include understanding how regions of given intensity interact in the

* Partially supported by a grant from the National Science Foundation.

0022-0396/95 $6.00

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course of the blurring. For example, \( u(x, t) \) represents the amount of heat at the point \( x \) at time \( t \). Since heat diffuses, we would expect that the region where the heat content locally exceeded a given amount would expand so that if two different regions joined they must merge and further expand and, moreover, no new regions of relative higher intensity would be created (see, e.g., [K] for a discussion of these questions). We shall see that both of these intuitive expectations are false; so we would like to understand locally what can happen.

Our goal then is to describe the local structure of mappings \( u: U \subset \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \) which are solutions to the heat equation. Since any other nonisotropic heat equation with constant coefficients can be reduced to (1) by a linear change of coordinates in \( \mathbb{R}^n \), the results will likewise be applicable for any constant coefficient heat equation.

If one considers one parameter families of functions on a manifold without regard to how they arose, then the local properties of such generic functions are described by Morse theory and the catastrophe theory of Thom [Th] for parametrized families (this includes an analysis of the generic creation and annihilation of critical points).

There are several reasons why this is insufficient for the case of solutions to the heat equation. First, it is not clear that generic solutions to the heat equation must be generic in the Morse theory sense. Second, standard local models for Morse singularities and their annihilation and creation do not satisfy the heat equation. How must these models be modified? Third, there is the question of what constitutes generic behavior. This depends upon what notion of local equivalence one uses between solutions to the heat equation.

If one restricts oneself to transformations that preserve the heat equation, then the classification is much too fine. Any quadratic function \( u(x) = \sum_{i=1}^{n} a_i x_i^2 \), where \( \sum_{i=1}^{n} a_i = 0 \) is a solution to the heat equation. However, the only linear transformations on \( \mathbb{R}^n \) that preserve the heat equation are orthogonal ones and for these the eigenvalues \( a_i \) are invariant. Even multiplying by a constant will only allow a change in the relative size of these values. Thus, the \( n \)-tuple \( (a_1, a_2, ..., a_n) \), up to a constant multiple, would determine a distinct class.

To obtain a more useful classification, we allow local (nonlinear) changes of coordinates in \( \mathbb{R}^{n+1} \) and apply this equivalence relation to solutions to the heat equation. Two extra considerations will reflect the special nature of the solutions: (1) keeping track of the blurring parameter when we change coordinates, and (2) keeping track of the relative intensities of the critical points to understand the changes in the isointensity surfaces. Unexpected behavior that isointensity surfaces can exhibit has been discovered by Lifshitz [L] for one case of Gaussian blurring; condition (2) would determine how much difference this makes in general.
We would like to apply singularity theory following the general lines established by Mather in his series of fundamental papers [M] and Martinet [Mar]. For example, the general methods of singularity theory have been applied to examples where distinguished parameters occur, such as imperfect bifurcation by Golubitsky and Schaeffer [G–S1, G–S2, G–S3] and "space-time unfoldings" by Wasserman [Wa1]. Unfortunately, the standard techniques of singularity theory cannot be directly invoked in the case at hand. Even though singularity theory can be applied under very general circumstances, its applicability for local questions depends upon having a certain "nice structure" on the space of germs being considered and upon the equivalence relation being given by the action of a group of germs of diffeomorphisms (see [D2]). However, the space of smooth germs of solutions to the heat equation does not have this nice structure, and the group of equivalences does not act on the space of solutions.

Nonetheless, the purpose of this paper is to show how techniques of singularity theory can still be applied to solve this problem. We describe the main features of the method we use.

(1) We introduce two local notions of equivalence and classify germs of solutions to the heat equation using these notions of equivalence. The first, \(H\)-equivalence (which is the weaker notion), captures the equivalence of families of mappings \(u(x, t)\) up to change of coordinates in \(\mathbb{R}^n\) parametrized by \(t\) (and allowing reparametrization of \(t\)); the second, \(IS\)-equivalence (for intensity-sensitive equivalence), allows a nonlinear change of coordinates in the intensity coordinate provided the local equivalence at a singular point preserves the intensity of that critical point. These notions of equivalence do not preserve the heat equation; however, they do preserve the structure of the space parametrized by time, the structure of the isointensity surfaces and how these change with time, and the relative intensities of the critical points for the second notion.

(2) The space of solutions fails to have the nice structure, which means that it is not a module in the appropriate generalized sense. Because of this, when we work at the jet level, we replace the standard use of monomials in representing germs by their power series by expanding solutions to the heat equation in terms of certain weighted homogeneous solutions. We show how to algebraically construct these solutions using an operator \(E\) that associates to a monomial a solution of the same weight containing that monomial as a term.

(3) To obtain genericity results, we apply transversality arguments, the weighted homogeneous decomposition from (2), and the infinitesimal algebraic criteria in singularity theory. Being able to use (2) to determine the structure of the space of solutions at the jet level allows us to modify
standard arguments from singularity theory relating the algebraic criteria and transversality conditions.

In Section 1 we state the genericity theorems. In Section 2 we give the local stable forms of the solutions to the heat equation that occur generically, and we indicate several of the unexpected consequences for Gaussian blurring that arise from the local classification. The analysis of the solutions using the weighted homogeneous decomposition is given in Section 3. In Sections 4 and 5 we give the proofs of the local classification and the genericity results.

The author is especially grateful to both Steve Pizer and John Gauch for introducing him to these questions and for their very helpful discussions.

1. The Genericity Theorems

We will introduce the two local notions of equivalence and state the genericity theorems for solutions to the heat equation using these notions of equivalence. First we note that the heat equation is invariant under translation, so it is enough to define the notions of equivalence for germs \( f: \mathbb{R}^{n+1}, 0 \to \mathbb{R}, 0 \). Also, we will be principally interested in the behavior for \( t > 0 \); since the solutions to the heat equation are smooth for \( t > 0 \), the local genericity results will be stated for smooth germs. Thus, unless there is an explicit statement to the contrary, all mappings and germs will be smooth.

The first notion, H-equivalence, takes into account the distinguished role of \( t \) by allowing change of coordinates in \( \mathbb{R}^n \) parametrized by \( t \), change of coordinates for \( t \), as well as translations in \( \mathbb{R} \) parametrized by \( t \). We will use local coordinates \((x, t) = (x_1, x_2, ..., x_n, t)\) for \( \mathbb{R}^{n+1} \) and \( y \) for \( \mathbb{R} \).

**Definition 1.1.** Germs \( f, g: \mathbb{R}^{n+1}, 0 \to \mathbb{R}, 0 \) will be said to be H-equivalent if there is a germ of a diffeomorphism \( \varphi: \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{n+1}, 0 \) of the form \( \varphi(x, t) = (\varphi_1(x, t), \varphi_2(t)) \) with \( \varphi_2'(0) > 0 \) and a germ \( c(t): \mathbb{R}, 0 \to \mathbb{R}, 0 \) so that

\[
g(x, t) = f \circ \varphi(x, t) + c(t).
\]

**Remark 1.2.** In fact, one might want to weaken the equivalence by allowing a time-dependent change of coordinates of \( \mathbb{R} \) (preserving the sense of orientation of \( \mathbb{R} \)) in place of the addition of the constant \( c(t) \). This will be incorporated into the definition of IS-equivalence. We also want this refined notion of equivalence to keep track of the local changes of an iso-intensity surface as it undergoes a transition and the intensity level of that critical point.
Definition 1.3. Germs $f, g : \mathbb{R}^{n+1}, 0 \to \mathbb{R}, 0$ will be said to be IS-equivalent if there are germs of diffeomorphisms $\varphi : \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{n+1}, 0$ of the form $\varphi(x, t) = (\varphi_1(x, t), \varphi_2(t))$ with $\varphi_2(0) > 0$ and $\psi : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ of the form $\psi(y, t) = (\psi_1(y, t), t)$ with $(\partial \psi_2/\partial t)(0, 0) > 0$ and $\psi(0, t) = 0$ for all $t$ and a constant $c$ so that

$$g(x, t) = \psi(f \circ \varphi(x, t), t) + c$$

(in fact, $c$ must be 0 but when we consider stability of the germs under deformations we wish to allow the target to move so $c$ may be nonzero for nonzero parameter values).

Remark. H-equivalence is the usual notion of $\mathcal{A}$-equivalence of unfoldings in the parlance of singularity theory, while IS-equivalence is given by "$\mathcal{A}$-equivalence of unfoldings preserving the target." As mentioned above, these notions of equivalence do not preserve the heat equation.

Genericity is described via the notions of stability and versality. Let $f(x, t, u) : \mathbb{R}^{n+1+q}, 0 \to \mathbb{R}, 0$ be a deformation of $f_0$, by which we mean that $f(x, t, 0) = f_0(x, t)$.

Definition 1.4. A germ $f_0 : \mathbb{R}^{n+1}, 0 \to \mathbb{R}, 0$ is $H$-stable if for any deformation $f : \mathbb{R}^{n+1+q}, 0 \to \mathbb{R}, 0$ of $f_0$, there is a germ of a diffeomorphism $\varphi : \mathbb{R}^{n+1+q}, 0 \to \mathbb{R}^{n+1+q}, 0$ of the form $\varphi(x, t, u) = (\varphi_1(x, t, u), \varphi_2(t, u), u)$ with $(\partial \varphi_2/\partial t)(0, 0) > 0$ and a germ $c(t, u) : \mathbb{R}^{1+q}, 0 \to \mathbb{R}, 0$ so that

$$f_0(x, t) = f \circ \varphi(x, t, u) + c(t, u).$$

Likewise the germ $f_0$ is IS-stable if for any deformation $f : \mathbb{R}^{n+1+q}, 0 \to \mathbb{R}, 0$ of $f_0$, there are germs of diffeomorphisms $\varphi : \mathbb{R}^{n+1+q}, 0 \to \mathbb{R}^{n+1+q}, 0$ of the form $\varphi(x, t, u) = (\varphi_1(x, t, u), \varphi_2(t, u), u)$ with $(\partial \varphi_2/\partial t)(0, 0) > 0$ and $\psi : \mathbb{R}^{2+q}, 0 \to \mathbb{R}^{2+q}, 0$ of the form $\psi(y, t, u) = (\psi_1(y, t, u), t, u)$ with $(\partial \psi_2/\partial y)(0, 0) > 0$ and $\psi(0, t, u) = 0$ for all $(t, u)$ and a germ $c(u) : \mathbb{R}^q, 0 \to \mathbb{R}, 0$ so that

$$f_0(x, t) = \psi(f \circ \varphi(x, t, u), t, u) + c(u).$$

Remark. The form of stability used here is stability under deformations. In the parlance of singularity theory, $f_0$ is H-stable (resp. IS-stable) if $f_0$ is its own versal unfolding, i.e., every small perturbation of $f_0$ is H-equivalent (resp. IS-equivalent) to $f_0$.

Let $U$ be an open subset of $\mathbb{R}^{n+1}$. 
DEFINITION 1.5. We say that \( f : U \to \mathbb{R} \) is \( H \)-generic (resp. \( IS \)-generic) if for each point \( (x_0, t_0) \in U \), the germ \( f : \mathbb{R}^{n+1}, (x_0, t_0) \to \mathbb{R}, f(x_0, t_0) \) is \( H \)-stable (resp. \( IS \)-stable).

Then the basic theorems describing the local structure of solutions to the heat equation are given by the following. We let \( \mathcal{H} \) denote the space of mappings in \( C^\infty(U, \mathbb{R}) \) which are solutions to the heat equation.

**THEOREM 1.** The set of mappings in \( C^\infty(U, \mathbb{R}) \) which are \( H \)-generic (respectively \( IS \)-generic) form a residual set for the Whitney \( C^\infty \)-topology. The set of mappings in \( \mathcal{H} \) which are \( H \)-generic (respectively \( IS \)-generic) form a residual set for the regular \( C^\infty \)-topology.

Recall that a residual set is the countable intersection of open dense sets. Both \( \mathcal{H} \) and \( C^\infty(U, \mathbb{R}) \) are Baire spaces for the Whitney \( C^k \)-topology or regular \( C^k \)-topology where \( 2 \leq k \leq \infty \) for \( \mathcal{H} \) (see, e.g., [G-G] and Section 5, Proposition 5.5 for \( \mathcal{H} \)). Thus, a residual set is still dense. Using the regular \( C^\infty \)-topology for \( \mathcal{H} \) basically states that given an arbitrarily large compact set, the set of mappings which are generic on this set forms an open dense subset.

The role of the initial "intensity function" in the genericity results is described by the following. Let \( U_1 \subset \mathbb{R}^a \) and \( 0 \in U_2 \subset \mathbb{R}_+ \) be open sets.

**THEOREM 2.** From among the set of functions \( u_0 \in C^k(U_1, \mathbb{R}) \), where \( 0 \leq k \leq \infty \), for which the heat equation has a solution \( u(x, t) \) on \( U_1 \times U_2 \) with initial condition \( u_0 \), there is a dense subset (in the regular \( C^k \)-topology) for which \( u \) is \( H \)-generic, respectively, \( IS \)-generic.

**THEOREM 3.** List I, respectively, list II, gives a complete list, up to the corresponding equivalence, for the sets of \( H \)-stable, respectively \( IS \)-stable, germs \( \mathbb{R}^{a+1}, 0 \to \mathbb{R}, 0 \).

If we consider problems in which other parameters are present, then finite codimension germs also become important, for they determine what occurs generically in parametrized families.

**THEOREM 4.** (1) The finite \( H \)-codimension germs consist of the stable ones (codimension = 0) together with those in list V.

(2) The finite \( IS \)-codimension germs which are unfoldings of \( A_1 \) and \( A_2 \) germs consist of the stable ones (codimension = 0) together with those in list V.

(3) Those germs of \( IS_c \)-codimension \( \leq 1 \) are the ones, other than stable ones, that appear generically in one parameter families of solutions of the
heat equation; these generic families are IS-equivalent to one of the IS-versal unfoldings appearing in list III.

(4) There is a topological redundancy in list III. The germs of type (2c) and (3c) are topologically IS-equivalent as families for \( \varepsilon = \pm 1 \), and these are topologically IS-equivalent to (2a) and (3a) in list IV. Furthermore, only type (4_{(3)}) appears for \( n = 2 \).

2. Classification of Germs for H-Equivalence and IS-Equivalence

In this section we give the classification of smooth germs of solutions to the heat equation up to H-equivalence and IS-equivalence. This classification includes:

1. the H-stable germs in list I;
2. the IS-stable germs in list II;
3. the finite H-codimension germs in list V and their IS-classification;
4. the versal unfoldings of IS\(_c\)-codimension 1 germs in list III; this list is topologically redundant as explained in list IV.

We also give some consequences of the classification in I and II for Gaussian Blurring in Examples 2.2–2.5.

In this classification we use an operator \( E \); that associates to a polynomial germ \( g \) another polynomial \( E(g) \), which is a solution to the heat equation (see Section 3 for the definition). In the lists the germ \( E(g) \) may be replaced by \( g \) without changing the H-equivalence or IS-equivalence class or the stability (although the germ will then fail to satisfy the heat equation).

**List I: H-Stable Germs**

(0) \[ u(x, t) = x_1 \] (submersion)

(1a) \[ \sum_{i=1}^{n} x_i^2 + (2n) t \]

(1b) \[ -\sum_{i=1}^{n} x_i^2 - (2n) t \]

(1c) \[ \sum_{i=1}^{n} a_i x_i^2, \quad \text{where } \sum_{i=1}^{n} a_i = 0, \quad \text{all } a_i \neq 0 \]

(any two having the same index as quadratic forms are H-equivalent)
(2a) \[ x_1^3 + 6tx_1 + Q(x_2, \ldots, x_n, t) \]

(3a) \[ x_1^3 - 6tx_1 - 6x_1x_2^2 + Q(x_2, \ldots, x_n, t) \]

with Q as in (1) except only depending on \( x_2, \ldots, x_n, t \).

List II: IS-stable Germs

(0), (1a), and (1b) are as in list I.

In (1c')–(3), if the quadratic forms \( \sum a_ix_i^2 \) have the same index and \( \sum a_i \) have the same sign then the germs are IS-equivalent.

(1c') \[ \sum_{i=1}^{n} a_ix_i^2 + 2\left(\sum_{i=1}^{n} a_i\right)t, \quad \sum_{i=1}^{n} a_i \neq 0, \quad \text{all } a_i \neq 0 \]

(1d) \[ \sum_{i=1}^{n} a_ix_i^2 + \varepsilon E(t^2), \quad \sum_{i=1}^{n} a_i = 0, \quad \text{all } a_i \neq 0, \quad \varepsilon = \pm 1 \]

(2b) \[ x_1^3 + 6tx_1 + \sum_{i=2}^{n} a_ix_i^2 + 2\left(\sum_{i=2}^{n} a_i\right)t, \quad \sum_{i=2}^{n} a_i \neq 0, \quad \text{all } a_i \neq 0 \]

(3b) \[ x_1^3 - 6tx_1 - 6x_1x_2^2 + \sum_{i=2}^{n} a_ix_i^2 + 2\left(\sum_{i=2}^{n} a_i\right)t, \quad \sum_{i=2}^{n} a_i \neq 0, \quad \text{all } a_i \neq 0 \]

Example 2.1. Consider the three germs \( 2x^2 - y^2 + 2t, \ x^2 - y^2, \ x^3 - 3xy^2 + x^2 - y^2 \). These germs are all H-equivalent and H-stable. However, for IS-equivalence the last two have infinite codimension. This essentially means that there are an infinite number of different ways to deform these germs so that the isointensity surfaces undergo (smoothly) distinct transitions. These last two germs can deform into the finite codimension germs of type \( (4_{(k)}) \) in list V. Also, note that the first germ is a local submersion at 0; however, it is neither H-equivalent nor IS-equivalent to the germ of type (0), which is also a submersion.

Before considering the classification of finite codimension germs we first examine some of the consequences of the classification of stable germs for the generic behavior of solutions to the heat equation.

Example 2.2 (Annihilation of critical points: \( x^3 + 6tx + y^2 + 2t \) is of type (2b) with \( n = 2 \)). This is the example considered by Lifshitz [L] for the annihilation of a pair of critical points. Both critical points must approach
the annihilation intensity level from the same side. In Morse theory the critical points may approach from opposite sides so that at a nearby level the isointensity curves evolve as in Fig. 1.

However, this cannot happen for solutions to the heat equation. Instead the curve surrounding the local maximum must break off into a separate component and then this component shrinks to a point and disappears (Fig. 2).

**Example 2.3** (Creation of critical points: $x^3 - 6tx - 6xy^2 + y^2 + 2t$ is of type (3b) with $n = 2$). It was expected based on the reasoning explained earlier that for solutions to the heat equation, no critical points would be created. However, Pizer and Gauch gave a heuristic argument for the possibility of critical points being created in certain circumstances. This example shows that such a creation of critical points not only can occur but, in fact, does so generically for solutions to the heat equation.

**Example 2.4** (Merging and separating of regions: $x^2 - y^2 + E(t^2)$ is of type (1d) with $n = 2$). Since isointensity curves separate regions of greater intensity from those of lower intensity, we would expect that if two regions bounded by isointensity curves of the same intensity level join then they will continue to grow together. In fact, this example shows that two such regions can join and then break apart again. This germ is given by the equation

$$x^2 - y^2 + t^2 + (1/2) t(x^2 + y^2) + (1/16)(x^2 + y^2)^2.$$
In fact, this germ is IS-equivalent to the simpler germ \( g(x, y, t) = x^2 - y^2 + t^2 \). Hence, we can see that the level curves for the intensity level 0 join together and then break apart again (Fig. 3).

This merging and separating of regions can become even more complicated for higher codimension germs which appear generically in families. We first list the codimension 1 germs.

One parameter families of solutions to the heat equation are IS-equivalent to versal unfoldings of germs of IS-codimension 1, where the extra parameter is denoted by \( v_1 \). The versal unfoldings are given below.

**List III: Versal Unfoldings for IS\(_c\)-Codimension 1 Germs**

For all of the following, \( \sum_{i=1}^{n} a_i = 0 \), all \( a_i \neq 0 \), \( \varepsilon = \pm 1 \).

\[
(4_{131}) \quad \sum_{i=1}^{n} a_i x_i^2 + \varepsilon E(t^3 + v_1 t)
\]

\[
(2c) \quad x_1^3 + 6tx_1 + \sum_{i=2}^{n} a_i x_i^2 + \varepsilon E(t^2 + v_1 t)
\]

\[
(3c) \quad x_1^3 - 6tx_1 - 6x_1 x_2^2 + \sum_{i=2}^{n} a_i x_i^2 + \varepsilon E(t^2 + v_1 t).
\]

Here \( v_1 \) is treated as a constant by the operator \( E \).

Furthermore, there is a topological redundancy in this classification. The IS-versal unfoldings for (2a) and (3a) are given by

\[
(2a) \quad x_1^3 + 6tx_1 + \sum_{i=2}^{n} a_i x_i^2 + E(v_2 t^2 + v_1 t)
\]

\[
(3a) \quad x_1^3 - 6tx_1 - 6x_1 x_2^2 + \sum_{i=2}^{n} a_i x_i^2 + E(v_2 t^2 + v_1 t).
\]
Both of these unfoldings are topologically IS-equivalent as unfoldings to the constant unfolding on the parameter \( v_2 \) of the germs in list V (where the topological equivalence is actually smooth off the \((t, v_1, v_2)\)-subspace).

**List IV:** Topologically IS-Versal Unfoldings of Topological IS\(_e\)-Codimension 1 Germs

\[
(2a) \quad x_1^3 + 6tx_1 + \sum_{i=2}^{n} a_ix_i^2 + E(v_1t)
\]

\[
(3a) \quad x_1^3 - 6tx_1 - 6x_1x_2^2 + \sum_{i=2}^{n} a_ix_i^2 + E(v_1t).
\]

**Remark.** Not only are the unfoldings in list V topologically equivalent to those for (2c) and (3c) in list IV, but the unfoldings for (2c) and (3c) in list IV for the different values of \( \varepsilon \) are topologically IS-equivalent as unfoldings, even though they are not IS-equivalent.

This same phenomena of topological redundancy among the versal unfoldings of simple germs in a classification has been discovered by Rieger for the classification of apparent contours of projections of surfaces for another problem in computer vision [R].

**Example 2.5** (Merging and separating of regions reconsidered). Note that the conditions on the \( a_\) prevent all but \((4,3)\) from occurring for \( n = 2 \). However, \((4,3)\) exhibits even more complicated behavior regarding the merging and separating of regions. When \( v_1 < 0 \) with \( \varepsilon = 1 \), the region of intensity \( > 0 \) for \( x^2 - y^2 + r_1^3 + v_1t \) undergoes a series of separations and mergers. With the intervals as indicated in the graph of \( r_1^3 + v_1t \), we see that in interval A there is a single region as indicated (Fig. 4). In interval B the region separates into two parts. These regions rejoin in C, only to undergo a final separation in D.

**List V:** Nonstable Finite H-Codimension Germs

The following list contains up to H-equivalence, all finite H-codimension germs. Such germs can be viewed as unfoldings on the parameter \( t \) of a germ of type \( A_1 \) or \( A_2 \). The classification is actually given up to IS-equivalence,
so that all germs of type \((4_{(k)})\) are H-equivalent to type \((1c)\), the germs of type (2) or (3) form a single H-equivalence class, and all of the types (5)–(8) are equivalent to a type \((5_{(m)})\) for the same value of \(m\) (or \(m = 2\ell\)). Again the quadratic forms \(\sum a_i x_i^2\) must have the same index for the germs to be either H- or IS-equivalent.

In all of the following, \(\sum a_i = 0\), all \(a_i \neq 0\), \(m \geq 2\), \(k \geq 2\) (except in \((4_{(k)})\), \(\ell \geq 1\), \(\varepsilon\), \(\gamma = \pm 1\).

\[ (4_{(k)}) \sum_{i=1}^{n} a_i x_i^2 + \varepsilon E(t^k), \quad k > 2 \]

\[ (2c) \quad x_1^2 + 6t x_1 + \sum_{i=2}^{n} a_i x_i^2 + \varepsilon E(t^2) \]

\[ (2a) \quad x_1^2 + 6t x_1 + \sum_{i=2}^{n} a_i x_i^2 \]

\[ (3c) \quad x_1^2 - 6t x_1 - 6x_1 x_2^2 + \sum_{i=3}^{n} a_i x_i^2 + \varepsilon E(t^2) \]

\[ (3a) \quad x_1^2 - 6t x_1 - 6x_1 x_2^2 + \sum_{i=3}^{n} a_i x_i^2 \]

\[ (5_{(m)}) \quad x_1^2 - 3x_1 x_2^2 + \gamma E(x_1 t^m) + \sum_{i=2}^{n} a_i x_i^2 \]

\[ (6_{(k,m)}) \quad x_1^2 - 3x_1 x_2^2 + \gamma E(x_1 t^m) + \sum_{i=2}^{n} a_i x_i^2 + \varepsilon E(t^k), \quad k < 3m, \quad 2k \neq 3m \]

\[ (7_{(l',k)}) \quad x_1^2 - 3x_1 x_2^2 + \gamma E(x_1 t^{2l'}) + \sum_{i=2}^{n} a_i x_i^2 + \varepsilon E(t^{2l'}) + \varepsilon E(t^{k}), \quad \text{either } 3l' < k < 6l' \quad \text{or} \quad k > 3l', \quad 4\gamma + 27x^2 = 0 \]

\[ (8_{(l)}) \quad x_1^2 - 3x_1 x_2^2 + \gamma E(x_1 t^{2l'}) + \sum_{i=2}^{n} a_i x_i^2 + \varepsilon E(t^{2l'}), \quad 4\gamma + 27x^2 \neq 0. \]

3. Decomposing Solutions of the Heat Equation by Weighted Homogeneity

To understand properties of solutions to the heat equation, we take a different tack from the usual approach of viewing solutions as arising via convolution with the Gaussian kernel. Instead, we consider a weight
decomposition of the space of polynomial germs in such a way that the
heat operator preserves the decomposition. This allows us to understand
the local properties of solutions to the heat equation in terms of the
weighted terms in their Taylor expansion. We consider smooth solutions;
since the heat equation is invariant under translation, we may assume we
are considering germs at the origin.

Let the heat operator \( A - \partial / \partial t \) be denoted by \( D \). Also, we assign weights
to \((x, t)\) so that \( \text{wt}(x_i) = 1 \) for all \( i \), and \( \text{wt}(t) = 2 \). We use the standard
notation for monomials \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and we let
\( |\alpha| = \sum_{i=1}^{n} \alpha_i \). Then, we define the weight of the monomial \( x^\alpha t^b \) to be
\( \text{wt}(x^\alpha t^b) = |\alpha| + 2b \). A polynomial is weighted homogeneous of weight \( m \) if
the monomials appearing in it all have weight \( m \). Let

\[
W_k = \{ \text{weighted homogeneous polynomials in } (x, t) \text{ of weight } k \}
\]

and also let

\[
\mathcal{P}_k = \{ \text{homogeneous polynomials in } x \text{ of degree } k \}.
\]

Then,

\[
W_k = \mathcal{P}_k \oplus t \cdot \mathcal{P}_{k-2} \oplus t^2 \cdot \mathcal{P}_{k-4} \oplus \cdots \oplus t^{[k/2]} \cdot \mathcal{P}_{k-2[k/2]}.
\] (3.1)

To describe the polynomial solutions to the heat equation we begin with
a very elementary observation that is key to the algebraic analysis (and
whose simple verification is left to the reader).

**Lemma 3.2.** (i) If \( u: \mathbb{R}^{n+1} \to \mathbb{R} \) has Taylor series \( u_1 \) and \( Du = 0 \) then
\( Du_1 = 0 \).

(ii) \( D \) preserves the decomposition by the subspaces \( W_k \) and decreases
weight by 2.

Thus, it suffices to examine the behavior of \( D \) on \( W_k \). The representation
of solutions to \( Du = 0 \) on \( W_k \) is given by the following.

**Lemma 3.3.** If \( g \in W_k \) satisfies \( Dg = 0 \) then for \( g_0(x) = g(x, 0) \)

\[
g(x, t) = \exp(tD)(g_0) = \sum_{j=1}^{[k/2]+1} \frac{t^j}{j!} D^j(g_0) \left( \frac{t^j}{j!} \right). \] (3.4)

**Proof.** This result follows analytically provided \( \exp(tD)(g_0) \) converges
because the heat equation defines a flow on the space of functions and this
is the standard method to integrate such a flow with initial condition.
However, we can also see how the algebraic decomposition (3.1) yields
the result. We see that in Fig. 5, \( \partial / \partial t \) acts along the vertical arrow and \( A \) acts
along the horizontal arrow, each preserving the decomposition (3.1). Thus, if \( g = \sum g_j(x)(t^j/j!) \) with \( \text{wt}(g_j) = k - 2j \) and \( D(g) = 0 \), then

\[
\Delta \left( g_j(x) \frac{t^j}{j!} \right) = \frac{\partial}{\partial t} \left( g_{j+1}(x) \frac{t^{j+1}}{(j+1)!} \right) \quad \text{or} \quad \Delta(g_j(x)) = g_{j+1}(x).
\]

Conversely, it is possible to begin with a germ \( g \) and add terms to it to obtain a solution to the heat equation. For this we recall basic properties of \( \Delta \) acting on polynomials in \( x \) (see, e.g., [F, Chap. 2]).

Let \( H_k = \ker(\Delta|\mathcal{P}_k) \); this is the space of harmonic polynomials of degree \( k \). Also, let \( r^2 = \sum_{i=1}^{n} x_i^2 \).

**Proposition 3.5.** (i) \( \Delta : \mathcal{P}_k \to \mathcal{P}_{k-2} \) is surjective for all \( k \);

(ii) \( \mathcal{P}_k = H_k \oplus r^2 \cdot \mathcal{P}_{k-2} \);

(iii) \( \mathcal{P}_k = H_k \oplus r^2 \cdot H_{k-2} \oplus r^4 \cdot H_{k-4} \cdots \oplus r^{2[k/2]} \cdot H_{k-2[k/2]} \).

For weight \( k \) the harmonic polynomials \( H_k \) are already solutions to the heat equation. We next wish to use Proposition 3.5 to construct a complementary subspace of solutions. Furthermore, given a monomial \( x^a(t^j/j!) \) we will construct a solution by adding to it terms of the same weight. Hence, the usual role of monomials in singularity theory will be replaced by these canonical solutions.

**Lemma 3.6.** There is a unique linear operator \( E \) on the polynomial ring \( \mathbb{R}[x, t] \) that preserves weights and satisfies:

(i) \( DE(g) = 0 \) for \( g \in \mathbb{R}[x, t] \);

(ii) \( E \left( x^a \cdot \frac{t^j}{j!} \right) = \sum_{i=0}^{\lfloor |x| \rfloor + 1} A'(x^a) \cdot \left( \frac{t^{j+i}}{(j+i)!} \right) \mod \left( \sum t^{j-m} r^{2m} \cdot W_{|a|} \right) \).

**Remark.** It follows from the proof that if \( g(x) \in W_k \) then \( E(g) \) is given by (3.4). As a consequence of this lemma we have
COROLLARY 3.7. If $g$ is a polynomial with at most one nonzero monomial of any given weight, then for any nonzero monomial of $g$, that monomial appears in $E(g)$ with the same coefficient.

Proof. By the linearity of $E$ it is enough to define it for monomials $x^a \cdot (t^j/j!)$ so that Lemma 3.6 (ii) is satisfied and then show that this uniquely determines $E$. Then, for this monomial, Lemma 3.3 shows us which terms with higher powers of $t$ to add. For lower powers of $t$ we must solve $A^h(u) = x^a$. However, by Proposition 3.5 there is a unique inverse to $A$ defined

$$A^{-1} : \mathcal{R}_{k+2} \rightarrow r^2 \cdot \mathcal{R}_{k+2}.$$ (3.8)

Note that this inverse is not multiplication by $r^2$. However, it is uniquely defined. By the algebraic proof we gave for Lemma 3.3 together with the decomposition (iii) in Proposition 3.5 it follows that the solution is uniquely determined.

The action of $A$ on $r^{2j} \cdot H_k$ has an especially simple form.

LEMMA 3.9. If $p \in H_k$ then

$$A(r^{2j} \cdot p) = c(j, k) \cdot r^{2j-2} \cdot p,$$ where $c(j, k) = 2j(n + 2(k + j - 1)).$

Proof. By direct computation

$$A(r^{2j} \cdot p) = A(r^{2j}) \cdot p + 2 \sum_{i=1}^{n} \frac{\partial r^{2j}}{\partial x_i} \cdot \frac{\partial p}{\partial x_i} + r^{2j} \cdot A(p)$$

or, keeping in mind that $p$ is harmonic,

$$A(r^{2j} \cdot p) = 2j(2j - 2 + n) \cdot r^{2j-2} \cdot p + 4j2^{j-2} \sum_{i=1}^{n} x_i \cdot \frac{\partial p}{\partial x_i} + 0.$$

By Euler's relation applied to the second term

$$A(r^{2j} \cdot p) = 2j(2j - 2 + n) \cdot r^{2j-2} \cdot p + 4jkr^{2j-2} \cdot p = c(j, k) \cdot r^{2j-2} \cdot p.$$ (3.9)

From this we can explicitly compute $E$ on the summands in (iii) of Proposition 3.5.

COROLLARY 3.10. If $p \in H_k$ then

$$E\left(r^{2j} \cdot p \frac{t^j}{j!}\right) = Q(r, t) \cdot p,$$
where

\[ Q(r, t) = \left( \sum_{m=0}^{\ell+j} a(j, k, m) \cdot r^{2(j+\ell-m)} \cdot \frac{t^m}{m!} \right) \]

and

\[ a(j, k, m) = \left( \prod_{i=1}^{m+j} c(\ell - i + 1, k) \right)^{\delta} \]

with \( \delta = \text{sign}(m-j) \) (and \( = 0 \) if \( j = m \)).

**Proof.** Immediate from Lemmas 3.6 and 3.9 since 3.9 gives the value of \( \Delta \) and \( \Delta^{-1} \) on the summands. \( \square \)

Furthermore, a basis for the solutions to the heat equation is given via Proposition 3.5 and Corollary 3.10.

**Corollary 3.11.** A basis for the subspace of solutions to the heat equation in \( W_m \) is given by \( \{ E(r^{2\ell} \cdot p_k \cdot) \} \), where \( 2\ell + k = m \) and \( p_k \) ranges over a basis for \( H_k \).

**Example 3.12.** We give for the lowest weights both generators for the harmonic polynomials and a complementary basis for the remaining solutions for \( \mathbb{R}^3 \) with coordinates \( (x, y, t) \).

<table>
<thead>
<tr>
<th>wt</th>
<th>( H_k )</th>
<th>Remaining basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( x, y )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( x^2 - y^2, xy )</td>
<td>( E(t) = t + 1/4 \cdot r^2 )</td>
</tr>
<tr>
<td>3</td>
<td>( x^3 - 3xy^2, y^3 - 3yx^2 )</td>
<td>( x^3 + 6tx, y^3 + 6ty ), or, e.g., ( E(xt) = xt + 1/8(xr^2), E(yt) )</td>
</tr>
<tr>
<td>4</td>
<td>( x^3y - xy^3, x^4 + y^4 - 6x^2y^2 )</td>
<td>( x^4 + 12tx^2, x^2y^2 + 2t \cdot r^2, xy^3 + 6txy, ) or, e.g., ( E((x^2 - y^2) t), E(xyt), ) and ( E(t^2) = t^2 + (1/2) t \cdot r^2 + (1/16) r^4 )</td>
</tr>
</tbody>
</table>

We will be using these bases in the next section when we classify the stable and finite codimension germs.

4. **Proof of the Classification**

In this section we will give the proofs of the classification results. Before proceeding we indicate the method that we will use. The main idea which comes from singularity theory is to realize that the equivalence relations we
consider on the germs arise via the actions of groups of germs of diffeomorphisms. In his fundamental series of papers [M], Mather showed that questions of stability, equivalence, and "determinacy" could be solved for certain basic notions of equivalence using infinitesimal methods, which yield criteria involving the tangent spaces to these infinite dimensional groups. The two main notions of equivalence we consider, H- and IS-equivalence, are also defined via the actions of groups (which we also denote by H and IS). Mather's method can be extended (see [D2]) so it applies quite generally to groups such as these. However, this method works when applied to the action on the space of all germs. These groups do not act on the space of solutions to the heat equation and the space of solutions does not have the correct algebraic structure needed to apply [D2].

Instead, we shall classify the germs which are solutions to the heat equation and which are stable within the space of all germs. It turns out that this describes locally all of the generic solutions. If we denote either of these groups by $\mathcal{G}$, then by the unfolding theorem (Theorem 9.3 of [D2]), $f$ is stable (under deformations) iff

\[ T \mathcal{G} \cdot f = T \mathcal{G}_{(n+1,1)} e, \tag{4.1} \]

where $T \mathcal{G} \cdot f$ denotes the "extended" tangent space to the $\mathcal{G}$-orbit of $f$, and $T \mathcal{G}_{(n+1,1)} e$ denotes the "extended" tangent space to the space of germs $\mathbb{R}^{n+1} \times \mathbb{R}, 0 \to \mathbb{R}$, 0. The tangent spaces to the groups and spaces of germs are modules over systems of rings; in fact, this is a key property which fails for the space of solutions.

This algebraic description allows us to deduce as a consequence of (4.1) that for a simpler group $\mathcal{G}^+$ which appears in the usual catastrophe theory, $f(x,0)$ has $\mathcal{G}^+$-codimension \( \leq 1 \). Such germs are completely known up to $\mathcal{G}^+$-equivalence. Then, we can begin classifying the germs satisfying (4.1) from among those satisfying the $\mathcal{G}^+$-codimension condition. Furthermore, we apply classification arguments similar to those of, e.g., Arnold [A] and Siersma [S] to solutions to the heat equation replacing the standard use of monomials by the weighted homogeneous solutions we have identified. In carrying out this classification, we make use of two standard techniques.

1. We use the splitting lemma, which is basically a parametrized form of the Morse lemma, to simplify the form of a germ once its quadratic terms are known; although this lemma was proven for $\mathcal{G}$-equivalence, it is also valid for both H- and IS-equivalence.

2. We use Mather's geometric lemma, which allows us to determine when a family of germs $g_u$ are all $\mathcal{G}$-equivalent. This criterion is again given infinitesimally in terms of the $\mathcal{G}$-codimension of $g_u$ and the infinitesimal condition $\partial g_u / \partial u \in T \mathcal{G} \cdot g_u$. 
To begin with the infinitesimal calculations we first establish some notation.

We use local coordinates $x$ for $\mathbb{R}^n$, $0$, $(x, t)$ for $\mathbb{R}^{n+1}$, $0$, and $y$ for $\mathbb{R}$, $0$. We denote the ring of smooth germs $\mathbb{R}^n$, $0 \to \mathbb{R}$ by $\mathcal{C}_x$, with maximal ideal $\mathcal{M}_x$; similarly, we denote the ring of smooth germs $\mathbb{R}^{n+1}$, $0 \to \mathbb{R}$ by $\mathcal{C}_{x,t}$, $\mathbb{R}$, $0 \to \mathbb{R}$ by $\mathcal{C}_y$, etc. Also, for $f: \mathbb{R}^{n+1}, 0 \to \mathbb{R}, 0$, the ring homomorphism $f^*: \mathcal{C}_y \to \mathcal{C}_{x,t}$, induced by composition $(f^*(h) = h \circ f)$, will be understood without always being explicitly stated. Given a ring $R$, the $R$-module generated by $\varphi_1, \ldots, \varphi_k$ will be denoted by $R\{\varphi_1, \ldots, \varphi_k\}$ or $R\{\varphi_i\}$ if $k$ is understood. These rings will be used to describe the algebraic structure of the tangent spaces.

The "extended" tangent space to $\mathcal{C}_{x_{\alpha+1,1}}$, the space of germs $\mathbb{R}^{n+1}$, $0 \to \mathbb{R}$, $0$, is isomorphic to the space of germs $\mathbb{R}^{n+1}$, $0 \to \mathbb{R}$, which is denoted by $\mathcal{C}_{x,t}$ (see, e.g., [D2a, Section 4]). To determine the extended tangent spaces to $H$ and $IS$ we must explicitly describe the action of $1$-parameter unfolding groups for these groups on $1$-parameter unfoldings of germs in $\mathcal{C}_{(\alpha+1,1)}$ and then compute the derivative of the orbit map for this action. The actions on the $1$-parameter unfoldings are essentially given by Definition 1.4.

The $1$-parameter unfolding group for $H$ consists of pairs $(\varphi, c)$, where $\varphi: \mathbb{R}^{n+2}, 0 \to \mathbb{R}^{n+2}, 0$ is a germ of a diffeomorphism of the form $\varphi(x, t, u) = (\varphi_1(x, t, u), \varphi_2(t, u), u)$ with $(\partial \varphi_2/\partial t)(0, 0) > 0$ and $c(t, u): \mathbb{R}^2, 0 \to \mathbb{R}, 0$ is a germ. The pair acts on a $1$-parameter unfolding $F(x,t,u) = (\tilde{F}(x, t, u), t, u)$ with $\tilde{F}(x, t, 0) = f(x, t)$ by

$$(\varphi, c) \cdot F(x, t, u) = F \circ \varphi(x, t, u) + (c(t, u), 0).$$

Similarly, the $1$-parameter unfolding group for $IS$ consists of triples $(\varphi, \psi, c)$, where $\varphi: \mathbb{R}^{n+2}, 0 \to \mathbb{R}^{n+2}, 0$ is a germ of diffeomorphism of the form $\varphi(x, t, u) = (\varphi_1(x, t, u), \varphi_2(t, u), u)$ with $(\partial \varphi_2/\partial t)(0, 0) > 0$ and $\psi: \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ of the form $\psi(y, t, u) = (\psi(y, t, u), t, u)$ with $(\partial \psi/\partial y)(0, 0, 0) > 0$ and $\psi(0, t, u) = 0$ for all $(t, u)$ and a germ $c(u): \mathbb{R}, 0 \to \mathbb{R}, 0$. The triple acts on a $1$-parameter unfolding $F(x, t, u) = (\tilde{F}(x, t, u), t, u)$ with $\tilde{F}(x, t, 0) = f(x, t)$ by

$$(\varphi, \psi, c) \cdot F(x, t, u) = \psi \circ F \circ \varphi(x, t, u) + (c(u), 0).$$

The extended tangent spaces to $H$- and $IS$-orbits are then computed by differentiating (4.2) and (4.3) with respect to $u$ and evaluating at $u = 0$ (see, e.g., [D2a, Section 4]). They are given by

$$TH_{x,t} \cdot f = \mathcal{C}_{x,t} \left\{ \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial t} \right\},$$

$$TIS_{x,t} \cdot f = \mathcal{C}_{x,t} \left\{ \frac{\partial f}{\partial x_i} \right\} + \mathcal{M}_x \cdot \mathcal{C}_{x,t} + \mathcal{C}_t \left\{ \frac{\partial f}{\partial t} \right\} + \langle 1 \rangle.$$
where \( \langle \varphi_1, \ldots, \varphi_k \rangle \) will denote the vector space generated by \( \varphi_1, \ldots, \varphi_k \) and \( m_y \cdot \mathcal{C}_x \) is mapped into \( \mathcal{C}_{x,t} \) by \( f^* \), where \( f^*(x, t) = (f(x, t), t) \).

\( \mathcal{H} \)-equivalence itself can be viewed as an equivalence (\( \mathcal{A}^+ \)-equivalence) for 1-parameter unfoldings of germs \( f_0: \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0 \), where \( t \) is viewed as the unfolding parameter. We likewise have the extended tangent space for \( \mathcal{A}^+ \)-equivalence:

\[
T \mathcal{A}^+_e \cdot f_0 = \mathcal{C}_x \left\{ \frac{\partial f_0}{\partial x_i} \right\} + \langle 1 \rangle.
\]

Now let \( \mathcal{G} \) denote either of the groups \( \mathcal{H} \) or IS. These groups acting on \( \mathcal{C}_{(n+1,1)} \) (together with their unfolding groups acting on unfoldings of these germs) are "geometric subgroups of \( \mathcal{A} \) or \( \mathcal{H} \)" in the sense of \([D2]\); all of the properties except the tangent space condition are immediate, while computations similar to those for the orbit spaces show that the tangent spaces are modules over the system of rings \( \mathcal{C}_t \rightarrow \mathcal{C}_{x,i} \) for \( \mathcal{H} \) or \( \mathcal{R} \rightarrow \mathcal{C}_t \rightarrow \mathcal{C}_{x,i} \rightarrow \mathcal{C}_{x,t} \) for IS (note that \( \mathcal{R} \) is the ring of smooth germs on \( \mathbb{R}^0 \)). This is an adequately ordered system of DA-algebras in the sense of \([D2]\), and the infinitesimal orbit map is a homomorphism of such modules so the tangent space condition is satisfied. Then, by the unfolding theorem (Theorem 9.3 of \([D2]\)) a germ \( f: \mathbb{R}^{n+1}, 0 \rightarrow \mathbb{R}, 0 \) is \( \mathcal{G} \)-stable (under deformations) iff

\[
T \mathcal{G}_e \cdot f = \mathcal{C}_{x,t} \quad (= T \mathcal{C}_{(n+1,1)}).
\]  \hspace{1cm} (4.4)

We can deduce a simple consequence even if we replace (4.4) by the weaker condition that \( f \) has finite \( \mathcal{G} \)-codimension, which means that

\[
\mathcal{G}_e \text{-} \text{codim}(f) = \dim_{\mathbb{R}}(\mathcal{C}_{x,t}/T \mathcal{G}_e \cdot f) < \infty \quad \text{(or for } \mathcal{H}^+, \mathcal{A}_e^+ \text{-} \text{codim}(f_0) = \dim_{\mathbb{R}}(\mathcal{C}_x/T \mathcal{A}_e^+ \cdot f_0) < \infty \).
\]

**Lemma 4.5.** (i) *If a germ \( f: \mathbb{R}^{n+1}, 0 \rightarrow \mathbb{R}, 0 \) has finite \( \mathcal{H} \)-codimension or is IS-stable and \( f_0(x) = f(x, 0) \), then \( \mathcal{A}_e^+ \)-codim(\( f_0 \)) \leq 1 \); moreover, if \( f \) is \( \mathcal{H} \)- or IS-stable then \((f(x, t), t)\) is an \( \mathcal{H}^+ \)-versal unfolding of \( f_0 \).

(ii) *If \( f_0 \) is weighted homogeneous and \( f \) has finite IS-codimension then

\[
\text{IS}_e \text{-} \text{codim}(f_0) \geq 2(\mathcal{A}_e^+ \text{-} \text{codim}(f_0) - 1).
\]

**Proof.** First consider the case of \( \mathcal{H} \)- or IS-stable germs; we divide (4.4) by \( m_y \cdot \mathcal{C}_x \) to obtain either

\[
\mathcal{C}_x \left\{ \frac{\partial f_0}{\partial x_i} \right\} + \left\langle 1, \left. \frac{\partial f}{\partial t} \right|_{t=0} \right\rangle = \mathcal{C}_x
\]  \hspace{1cm} (4.6)
or

\[ \mathcal{C}_x \left\{ \frac{\partial f_0}{\partial x_i} \right\} + m \varphi + \left\{ 1, \frac{\partial f}{\partial t} \bigg|_{t-0} \right\} = \mathcal{C}_x. \tag{4.7} \]

Then, (4.6) implies that \( \mathcal{R}^*_\mathcal{C}_x - \text{codim}(f_0) \leq 1 \), while (4.7) implies that \( \mathcal{H}^- \text{-codim}(f_0) \leq 2 \) (where \( \mathcal{H}^- \) denotes the group of contact equivalence (see [MIII])). However, by the classification of such germs [MV1], \( \mathcal{H}^- \text{-codim}(f_0) \leq 2 \) implies that \( f_0 \) is weighted homogeneous for some choice of weights so \( f_0^* : m \varphi \in \mathcal{C}_x \{ \partial f_0 / \partial x_i \} \) and hence we again obtain (4.6).

Then, again by the unfolding theorem, (4.6) is the infinitesimal criterion for the unfolding \( (f(x, t), t) \) to be \( \mathcal{R}^+ \)-versal.

Second, consider the case of finite \( H^- \) or IS-codimension. By the finite determinacy theorem for these groups, Theorem 10.2 of [D2b], \( f \) is \( H^- \) or IS-equivalent, as appropriate, to a polynomial germ. Hence, we can work within the category of analytic germs and the rings will denote the rings of analytic germs. Again dividing by \( m \varphi \), \( \mathcal{C}_x, t \) we obtain from

\[ \mathcal{T} \mathcal{C}_x : f + \langle \varphi_1, ..., \varphi_m \rangle = \mathcal{C}_x, t \]

that \( f_0 \) has finite \( \mathcal{R}^+ \)-codimension with complement to \( \mathcal{T} \mathcal{R}^+ : f \) spanned by \( \langle (\partial f / \partial t) \bigg|_{t=0}, \varphi_1, ..., \varphi_m \rangle \). Thus, \( f_0 \) has an algebraically isolated singularity at 0; therefore, \( \left\{ (\partial f_0 / \partial x_1), ..., (\partial f_0 / \partial x_n) \right\} \) forms a regular sequence in \( \mathcal{C}_x \). If \( I = \psi_0, \psi_1, ..., \psi_m \) form a basis for \( \mathcal{C}_x : \mathcal{C}_x \{ \partial f_0 / \partial x_i \} \), then \( \mathcal{C}_x : \mathcal{C}_x \{ \partial f / \partial x_i \} \) is a free \( \mathcal{C}_x \)-module generated by \( \{ \psi_0, \psi_1, ..., \psi_m \} \).

Then in the case of \( H^- \) equivalence, the finite codimension implies that \( L = \mathcal{C}_x : \mathcal{C}_x \{ \partial f / \partial x_i \} \) has the \( \mathcal{C}_x \)-submodule spanned by \( \{ 1, (\partial f / \partial t) \bigg|_{t=0} \} \), and the quotient by this submodule is a finite dimensional vector space. Hence, as a \( \mathcal{C}_x \)-module, \( \text{rank}(L) \leq 2 \). Thus, \( m + 1 \leq 2 \), implying \( \mathcal{R}^+_\mathcal{C}_x - \text{codim}(f_0) \leq 1 \).

For the case of IS-equivalence, \( f_0 \) is weighted homogeneous and we may choose the \( \psi_i \) to be weighted homogeneous. Also, \( f_0 + \sum_{i=0}^m u_i \psi_i \) is an \( \mathcal{R}^- \)-versal unfolding of \( f_0 \) [Wa2], so by the properties of versal unfoldings, \( f \) is equivalent as an \( \mathcal{R}^- \)-unfolding to \( f_0 + \sum_{i=0}^m u_i(t) \psi_i \), where \( u_i(0) = 0 \) for all \( i \) so \( u_i(t) \in m \). Since equivalence of \( \mathcal{R}^- \)-unfoldings is an IS-equivalence, we may assume \( f \) itself has this form. We now claim that

\[ f^*(y) = f \in \mathcal{C}_x, t \left\{ \frac{\partial f}{\partial x_i} \right\} + m \{ \psi_i \}. \tag{4.8} \]

To see this we use the Euler relation; let \( \text{wt}(x_i) = a_i, \text{wt}(\psi_i) = w_i, \) and \( \text{wt}(f_0) = d \), then

\[ \sum a_i x_i \frac{\partial f}{\partial x_i} = \sum a_i x_i \frac{\partial f_0}{\partial x_i} + \sum u_i(t) \left( \sum a_i x_i \frac{\partial \psi_i}{\partial x_i} \right) \]
or
\[
\sum a_i x_i \frac{\partial f}{\partial x_i} = d \left( f_0 + \sum u_i \psi_i \right) + \sum (w_j - d) u_j(t) \psi_j
\]

implying (4.8). Then (4.8) implies
\[
M \left( = \mathcal{C}_{x,t} \left\{ \frac{\partial f}{\partial x_i} \right\} + m_i m_j \mathcal{C}_{x,t} + m_i^2 \frac{\partial f}{\partial t} \right) \\
\subseteq \mathcal{C}_{x,t} \left\{ \frac{\partial f}{\partial x_i} \right\} + m_i^2 \{ \psi_j \}.
\]

Since \( \{ \psi_0, \psi_1, ..., \psi_m, t \psi_0, t \psi_1, ..., t \psi_m \} \) is a basis for \( \mathcal{C}_{x,t} \left\{ \frac{\partial f}{\partial x_i} \right\} + m_i^2 \{ \psi_j \} \) and \( \{1, f, \frac{\partial f}{\partial t}, t(\frac{\partial f}{\partial t})\} \) spans \( T \text{IS}_e \cdot f / M \), we conclude
\[
\dim_{\mathbb{R}} (\mathcal{C}_{x,t} / T \text{IS}_e \cdot f) \geq 2(m + 1) - 4 = 2(\mathcal{R}_e^+ - \text{codim}(f_0) - 1).
\]

**Remark.** By the preparation theorem, (4.6) implies that \( f \) is H-stable; however, neither (4.6) nor (4.7) implies that \( f \) is IS-stable.

By the classification of Thom [Th] and Arnold [A], the germs \( f_0 \) of \( \mathcal{R}_e^+ - \text{codim} \leq 1 \) are \( \mathcal{R}_e^+ \)-equivalent to either
\[
\sum_{i=1}^{n} \varepsilon_i x_i^2 \quad \text{or} \quad x_1^3 + \sum_{i=2}^{n} \varepsilon_i x_i^2 \quad \text{(all } \varepsilon_i = \pm 1).\]

We will reconsider the beginning of the classification of simple germs [A] or [S] for \( \mathcal{R}_e^+ \)-equivalence, except that we use the basic weighted homogeneous solutions to the heat equation, keeping track of the weights, working successively in \( W_1, W_2, \) etc.

The first reduction involves a version of the "splitting lemma."

**Lemma 4.10** (splitting lemma). *Let \( f : \mathbb{R}^{n+1} \to \mathbb{R}, 0 \) be a germ satisfying \( d_x f(0) = 0 \). Suppose \( d_x^2 f(0) \) has rank \( k \) and \( d_x^2 f(0) \mid \mathbb{R}^{n-k} \times \mathbb{R}^n \equiv 0 \). If \( f \) has finite H (resp. IS)-codimension, then \( f \) is H-equivalent (resp. IS-equivalent) to a germ of the form
\[
\sum_{i=1}^{k} \varepsilon_i x_i^2 + h(x_{k+1}, ..., x_n, t) \quad \text{with} \quad d_x h(0), d_x^2 h(0) \equiv 0
\]

(here all \( \varepsilon_i = \pm 1 \)).

**Remark.** Note that \( h \) given by the lemma is allowed to have a linear term involving \( t \) as well as quadratic terms involving \( tx_i, k+1 \leq i \leq n \), or \( t^2 \). This will be useful in the classification.
The proof will be postponed until the end of the section.

Second, to carry out the classification, we will use Mather’s geometric lemma. Again, if \(\mathcal{G}\) denotes either of the equivalence groups \(H\) or IS, it has an induced action on the \(\ell\)-jets \(J^\ell(n+1,1) \cong m_{x,i} / m_{x,i}^{\ell + 1}\). This action is a Lie group action; we can compute the tangent space to the orbit on the \(\ell\)-jet of \(f\), denoted \(T \mathcal{G} \cdot f\), to be

\[
T \mathcal{G} \cdot f = m_{x,i} \left\{ \frac{\partial f}{\partial x_i} \right\} + m_t \left\{ \frac{\partial f}{\partial t} \right\} \mod m_{x,i}^{\ell + 1}
\]

\[
T \text{IS} \cdot f = m_{x,i} \left\{ \frac{\partial f}{\partial x_i} \right\} + m_t \cdot \mathcal{C}_{x,i} + m_t \left\{ \frac{\partial f}{\partial t} \right\} \mod m_{x,i}^{\ell + 1}.
\]

Although Mather’s geometric lemma applies to arbitrary Lie group actions (see [MIII, Lemma 3.1]), we shall use it as it applies to the groups \(H^\ell\) or IS acting on \(\ell\)-jets.

**Lemma 4.11** (Mather’s geometric lemma). Let \(f_u : \mathbb{R}^{n+1}, 0 \to \mathbb{R}, 0\) be a family of germs defined for \(u \in I\), an interval. For \(\mathcal{G}\) denoting \(H\) or IS, if

(i) \(\dim_r T \mathcal{G} \cdot f_u\) is constant independent of \(u\) and

(ii) \(\partial f_u / \partial u \in T \mathcal{G} \cdot f_u\) for all \(u \in I\), then all \(f_u\) belong to the same \(\mathcal{G}\)-orbit. In particular, if \(f_{u_0}\) is \(\mathcal{G}\)-determined at order \(\ell\) then all \(f_u\) are \(\mathcal{G}\)-equivalent.

Armed with Lemmas 4.10 and 4.11 we can now carry out the classification.

(0) First suppose that \(f\) contains a linear term in \(x\). Then, \(\partial f / \partial x_i\) is a unit in \(\mathcal{C}_{x,i}\) for some \(i\); thus,

\[
\mathcal{C}_{x,i} \left\{ \frac{\partial f}{\partial x_i} \right\} = \mathcal{C}_{x,i} \quad \text{and} \quad m_{x,i} \left\{ \frac{\partial f}{\partial x_i} \right\} = m_{x,i},
\]

so \(f\) is both \(H\) and IS-stable and by Mather’s geometric lemma any two such germs are \(H\) and IS-equivalent because they can be joined by a path of such germs.

(1) Next suppose \(f\) has no linear terms in \(x\). Then, we consider the weight 2 terms. Suppose first that \(a_i^\ell f(0)\) has rank \(n\). Then, by an orthogonal transformation in \(\mathbb{R}^n\), we may assume these terms have the form \(\sum_{i=1}^n a_i x_i^2 + 2ct\). Here \(c = \sum_{i=1}^n a_i\) since this must be a solution to the heat equation, and all \(a_i \neq 0\) by assumption. This is \(H\)-stable and \(T H^\ell \cdot f = m_x m_{x,i} + m_t = m_{x,i}^2 + m_t\) independent of \(c\). Any two such germs for which the quadratic parts have the same index can be connected by a path of
such germs and so are H-equivalent; this gives (1a)–(c). For IS-equivalence, the weight 2 term does not even have finite IS-codimension if $c = 0$. However, if $c \neq 0$, then the weight 2 term is IS-stable and any two such germs with the same index and having $c$ of the same sign can be joined by a path of such germs; this also gives (1c'). If $c = 0$, we apply the splitting lemma to remove weight 3 terms, which must involve $x_i$'s, to obtain a germ whose weight 4 terms are of the form $b t^2 + \cdots$. If $b \neq 0$ then a calculation shows that this germ is IS-stable and $\Gamma_1 S' \cdot f = m_{r, t}^2$, independent of $b$.

Thus, two such germs with the same index for the quadratic form and the same sign for $b$ can be joined by a path of such germs and so are IS-equivalent. In particular the germs

$$\sum_{i=1}^{n} a_i x_i^2 + \varepsilon E(t^2), \quad \sum_{i=1}^{n} a_i = 0, \quad \text{all } a_i \neq 0, \quad \varepsilon = \pm 1$$

contain all of the equivalence classes and are solutions to the heat equation. These give (1d). If $b = 0$ then a calculation shows that the germ is not IS-stable.

(2 and 3) Next suppose $f$ has no linear terms in $x$ and that $d_x^2 f(0)$ has rank $n - 1$. Then, by an orthogonal transformation in $\mathbb{R}^n$, we may assume the weight 2 terms have the form

$$\sum_{i=2}^{n} a_i x_i^2 + 2ct, \quad \text{with} \quad c = \sum_{i=2}^{n} a_i, \quad \text{all } a_i \neq 0.$$  

Then, consider the weight 3 terms. They include $a x_1^3 + b t x_1 + \cdots$. If $a = 0$, then $f_0(x) (= f(x, 0))$ has $\mathcal{R}^+$-codim $\geq 2$. Then, by Lemma 4.5 $f$ has infinite H-codimension and IS$_x$-codimension $\geq 2$. Thus, we may suppose $a \neq 0$. Likewise if $b = 0$ then $(f(x, t), t)$ is not an $\mathcal{R}^+$-versal unfolding of $f_0$, and again by Lemma 4.5 $f$ is neither H- nor IS-stable. We may also assume $b \neq 0$. Again a calculation shows that such a germ is H-stable and $\mathcal{H}^f \cdot f = m_{x, t} m_{x, t} + m_t$ for $x' = (x_2, \ldots, x_n)$ independent of $(a, b, c)$ provided $a \neq 0, b \neq 0$. Thus, for a fixed index for the quadratic terms, and provided the signs of $a$ and $b$ do not change, two such germs can be joined by a path and so are H-equivalent. Multiplying $x_1$ and $t$ by appropriate positive constants we may obtain $\varepsilon_1 x_1^3 + \varepsilon_2 t x_1 + \cdots$ with $\varepsilon_i = \pm 1$. Further replacing $x_1$ by $-x_1$ shows that $(\varepsilon_1, \varepsilon_2)$ and $(-\varepsilon_1, -\varepsilon_2)$ yield equivalent germs. The germs (2a and 3a) in list I are H-equivalent to such germs and occur as solutions to the heat equation.

For IS-equivalence, the situation is slightly more delicate, for if $c = 0$ then the germ is not IS-stable. However, if $c \neq 0$, and $a, b \neq 0$, then it is IS-stable and the rest of the argument proceeds as already described. By Lemma 4.5 this completes the classification of the H- and IS-stable germs.
Classification of Finite H-Codimension Germs

For list V of finite H-codimension germs, other than the stable ones, we begin at step 1, but we also consider the classification up to IS-equivalence.

First, \( c = \sum_{i=1}^{n} a_i = 0 \). We already know the classification up to H-equivalence. By the splitting lemma, \( f \) is IS-equivalent to a germ \( f_i = \sum_{i=1}^{n} a_i x_i^2 + g(t) \) and \( g \) has no linear nor quadratic term. If \( bt^k \) is the smallest term appearing in \( g \) then we can make a change of coordinates in \( t \) preserving the positive \( t \)-direction to replace \( g(t) \) by \( \varepsilon t^k \) where \( \varepsilon = \text{sign}(b) \).

This yields (4.14).

For (2) and (3), when \( c = 0 \), we can again use the splitting lemma to obtain that \( f \) is IS-equivalent to a germ \( f_i \) of the form \( f_i = \sum_{i=2}^{n} a_i x_i^2 + g(x, t) \), where \( g \) has no terms of weight \( <3 \). First, \( f_i(x) = f_i(x, 0) \) is \( \mathcal{A} \)-equivalent to \( A_2 \) so that by the same versality argument given in the proof of Lemma 4.5, \( f_i \) is \( \mathcal{A} \)-equivalent as an unfolding to

\[
x_i^3 + u_1(t) x_i + u_2(t) + Q(x_2, ..., x_n),
\]

where \( Q(x_2, ..., x_n) = \sum_{i=2}^{n} a_i x_i^2 \) (all \( a_i \neq 0 \)). \( \mathcal{A} \)-equivalence of unfoldings is both an H- and an IS-equivalence. If \( u_i(t) = bt^m + \), then a change of coordinates in \( t \) preserving the positive \( t \)-direction allows us to write \( u_i(t) = \varepsilon t^m \) with \( \varepsilon = \pm 1 \). Note that by the definition of H-equivalence, (4.12) is H-equivalent to \( x_i^3 + \varepsilon t^m x_i + Q(x_2, ..., x_n) \).

Then, Mather's lemma shows that this is H-equivalent to \( x_i^3 + \varepsilon t^m x_i + Q(x_2, ..., x_n) \), giving the last claim for list V.

Finally, we must complete the classification of these germs using IS-equivalence. Let \( u_2(t) = ct^k + \). Then,

\[
3x_i^2 = -\varepsilon t^m \mod \mathcal{C}_{x_i,t} \begin{pmatrix} \frac{\partial f}{\partial x_i} \end{pmatrix},
\]

which implies

\[
f^2 = -(4\varepsilon/27) t^{3m} + (4\varepsilon/3) t^{m+k} x_i + c^2 t^{2k} + \cdots \mod \mathcal{C}_{x_i,t} \begin{pmatrix} \frac{\partial f}{\partial x_i} \end{pmatrix}. \tag{4.13}
\]

On the other hand, if \( 3m \neq 2k \) then the linear independence of \( f, \partial f/\partial x_1, \) and \( \partial f/\partial t \) implies that \( \mathcal{C}_{x_1,t} \{ x_1^3, t^m x_1, t^k \} \subset T IS \cdot f \).

If \( k > 3m \) then (4.13) together with Mather's lemma implies that \( f \) is IS-equivalent to (5). If \( k < 3m \) and \( 3m \neq 2k \) then again Mather's lemma implies that \( f \) is IS-equivalent to (6). For these cases when \( m = 1 \) we obtain germs IS-equivalent to (2a) or (2c) and (3a) or (3c). When \( 3m = 2k \) (\( = 6\ell \)),

\[
\tag{4.14}
\]

\[
3x_1^2 = -\varepsilon t^m \mod \mathcal{C}_{x_1,t} \begin{pmatrix} \frac{\partial f}{\partial x_1} \end{pmatrix},
\]

which implies

\[
f^2 = -(4\varepsilon/27) t^{3m} + (4\varepsilon/3) t^{m+k} x_1 + c^2 t^{2k} + \cdots \mod \mathcal{C}_{x_1,t} \begin{pmatrix} \frac{\partial f}{\partial x_1} \end{pmatrix}. \tag{4.13}
\]
the lowest weight parts of the three terms are no longer linearly independent because of the Euler relation. Also, (4.13) becomes

\[ f^2 = (-4e/27 + c^2) t^{6r} + (4e/3) t^{s' r} x_1 \cdots \mod m_{*, t} \left\{ \frac{\partial f}{\partial x_i} \right\}. \]  

(4.14)

The lowest weight terms will be linearly independent from the lowest weight terms of \( t^{k+1} (\partial f/\partial t) \) provided \( c^2 + 4e/27 \neq 0 \). Thus, provided \( u_2(t) = ct^{3'} + c't' + \cdots \) with \( r \geq 6' \) Mather’s lemma implies that \( f \) is IS-equivalent to (8). While if \( c^2 + 4e/27 = 0 \), and \( u_2(t) = ct^{3'} + c't' + \cdots \), then the Euler relation implies that \( t' \in T \cdot f \) so \( f \) is IS-equivalent to (7).

That the germs in list III are those of IS_{r'}-codimension 1 follows from Lemma 4.5 and the classification. That the versal unfoldings are as indicated follows by applying the infinitesimal criterion of the unfolding theorem.

**Topological Redundancy**

Finally, we establish the assertion of topological redundancy. We consider the case for (2); that for (3) is similar. The IS-versal unfolding of (2a) is given by

\[ x_1^3 + 6tx_1 + \sum_{i=2}^{n} a_i x_i^2 + E(v_2 t^2 + v_1 t). \]  

(4.15)

We wish to prove that this is topologically IS-equivalent to the constant unfolding on \( v_2 \) of

\[ x_1^3 + 6tx_1 + \sum_{i=2}^{n} a_i x_i^2 + E(v_1 t). \]  

(4.16)

We assign weights \( \text{wt}(x_1) = 2, \text{wt}(t) = 4 \), and \( \text{wt}(x_i) = 3 \) for \( i > 1 \), \( \text{wt}(v_1) = 2 \) and \( \text{wt}(v_2) = -2 \). Then the weight 6 part of (4.15) has the form

\[ x_1^3 + 6tx_1 + \sum_{i=2}^{n} a_i x_i^2 + (v_2 t^2 + v_1 t) \]

and likewise those for (4.16) are

\[ \tilde{f}(x, t, v_1) = x_1^3 + 6tx_1 + \sum_{i=2}^{n} a_i x_i^2 + v_1 t. \]  

(4.17)

The unfolding \( f(x, t, v_1) = (\tilde{f}(x, t, v_1), v_1) \) is the negative versal unfolding of \( \tilde{f}(x, t, 0) \). To prove the result, it is enough by [D3, Theorem 4] to prove that \( f \) has finite IS-codimension as an unfolding and then (4.16) defines an unfolding of finite graded IS-codimension.
We consider the Euler relation
\[
e(\hat{f}) = 6\hat{f} - 2x_1 \cdot \frac{\partial \hat{f}}{\partial x_1} - 3 \sum_{i=2}^{n} x_i \cdot \frac{\partial \hat{f}}{\partial x_i} - 4t \cdot \frac{\partial \hat{f}}{\partial t} = 2v_1 t.
\]

Then, \( t \cdot e \in \ker(\tau_0) \) in the notation of [D3, Section 4] and \( t \cdot e(\hat{f}) = 2v_1 t^2 \), which by Theorem 4 of [D3] implies that \( f \) has finite codimension as required.

Finally, the proof of the classification will be completed by proving the splitting lemma.

**Proof of Lemma 4.10.** We consider a standard type of proof of the result in the case of \( \mathcal{R}^{+} \)-equivalence and keep track of the equivalences at each step to see that they are, in fact, H- or IS-equivalences. The proof is by induction on \( \ell \) that \( f \) is H-equivalent (resp. IS-equivalent) to a germ of the form
\[
\sum_{i=1}^{k} \varepsilon_i x_i^2 + h_{\ell-1}(x_{k+1}, \ldots, x_n, t) \mod m_{x_1,t}^{\ell}, \tag{4.18}
\]
where \( h_{\ell-1} \) is a polynomial of degree \( \ell - 1 \) with no terms of degree \( < 3 \) except those involving \( tx_i, k + 1 \leq i \leq n, t, \) or \( t^2 \).

We can begin with an orthogonal change of coordinates to write the quadratic terms of \( f \) which only involve the \( x_i \) as \( Q(x_1, \ldots, x_k) = \sum_{i=1}^{k} \varepsilon_i x_i^2 \). Terms such as \( tx_i, 1 \leq i \leq k, \) can be removed by a change of coordinates of the form \( x_i' = x_i + a_i t, 1 \leq i \leq k \). This is a legitimate H- or IS-equivalence so the result follows for \( \ell = 3 \). Inductively, we show we may obtain the desired form without altering the terms of degree \( \leq \ell - 1 \) for \( \ell \geq 3 \). We may as well assume that \( f \) itself has the form (4.18). We may write
\[
f = Q(x_1, \ldots, x_k) + h_{\ell-1}(x_{k+1}, \ldots, x_n, t) + \psi_f(x_1, \ldots, x_n, t).
\]

Then,
\[
\psi_f(x_1, \ldots, x_n, t) = \sum_{i=1}^{k} x_i \cdot g_i(x_1, \ldots, x_n, t) + h'_{\ell}(x_{k+1}, \ldots, x_n, t) + \psi'(x_1, \ldots, x_n, t),
\]
where \( \deg(g_i) = \ell - 1 \geq 2, \deg(h'_{\ell}) = \ell, \) and \( \deg(\psi') > \ell \). Then, we change coordinates so that
\[
x_i' = x_i + (c_i/2) g_i(x_{i+1}, \ldots, x_n, t) \quad 1 \leq i \leq k
\]
and the other \( x_i \) remain unchanged. As \( \deg(g_i^2) = 2\ell - 2 \geq \ell + 1 \), (4.18) holds for \( \ell + 1 \). Also this is a legitimate H- or IS-equivalence.
Finally, as $f$ has finite H- or IS-codimension, by the finite determinacy theorem for these groups [D2], it is finitely H- or IS-determined, as is any germ H- or IS-equivalent to it, and the order of determinacy is the same, say $L$. Then any germ of the form (4.9) for $\ell = L$ is H- (resp. IS-) equivalent to

$$\sum_{i=1}^{k} e_i x_i^2 + h_{L-1}(x_{k+1}, \ldots, x_{\ell}, t).$$

5. Genericity via Transversality and Versality

In this section we give the proofs of Theorems 1 and 2. These proofs will rely on modifications of the Thom transversality theorem and a theorem of Mather establishing the equivalence of stability of germs and their infinitesimal versality [MV]. We begin by considering the $\ell$-jet space $J'(n+1,1)$. We again let $\mathcal{H}$ denote the space of solutions to the heat equation in $C^\infty(U, \mathbb{R})$, and we let $\mathcal{H}'$ denote the $\ell$-jets at the origin of solutions to the heat equation (without constant term). Then, we let $\mathcal{H}'(U) = U \times \mathcal{H}' \times \mathbb{R}$ denote the corresponding subbundle in the jet space $J'(U, \mathbb{R})$ consisting of $\ell$-jets of solutions to the heat equation. Then, for $f \in \mathcal{H}'(U)$, the $\ell$-jet extension map factors

$$j'(f): U \to \mathcal{H}'(U) \subseteq J'(U, \mathbb{R}).$$

Then the first result is a version of the Thom transversality theorem for the space $\mathcal{H}$.

**Theorem 5.1.** Suppose that $\mathcal{W}$ is a smooth submanifold of $\mathcal{H}'(U)$, then the set of mappings $f \in \mathcal{H}$ that are transverse to $\mathcal{W}$ (i.e., for which $j'(f)$ is transverse to $\mathcal{W}$) form a residual set for the regular $C^k$-topology ($\ell + 1 \leq k \leq \infty$).

The submanifolds we consider will often arise from submanifolds $\mathcal{U}$ of the $\ell$-jet space $J'(n+1,1)$. Then, we let $\mathcal{U}(U) = U \times \mathcal{H} \times \mathbb{R}$ denote the corresponding submanifold in the jet space $J'(U, \mathbb{R})$. For example, if $\mathcal{C}$ is an H- or IS-orbit ($= H \cdot f$ or $IS \cdot f$), we let $\mathcal{C}'$ denote the corresponding orbit in the $\ell$-jet space ($= H' \cdot f$ or $IS' \cdot f$). The set of $\ell$-jets of diffeomorphisms for either of these actions is an algebraic Lie group acting algebraically on $J'(n+1,1)$ so the orbit is a smooth submanifold; hence, $\mathcal{C}'(U)$ is smooth.

**Definition 5.2.** If $\mathcal{U} \subseteq J'(n+1,1)$ is a smooth submanifold, we will say that $\mathcal{H}$ is transverse to $\mathcal{U}$ if $\mathcal{H}'$ is transverse to $\mathcal{U}$ in $J'(n+1,1)$. We
also say that $\mathcal{H}$ is transverse to the orbit $\mathcal{O}$ if $\mathcal{H}$ is transverse to $\mathcal{O}^\ell$ for all $\ell \geq 0$.

Then, we deduce as a consequence an alternate version of the Thom transversality theorem for the space $\mathcal{H}$.

**Corollary 5.3.** Suppose that $\mathcal{H}$ is transverse to $\mathcal{Y}$, then the set of mappings in $\mathcal{H}$ that are transverse to $\mathcal{Y}(U)$ form a residual set for the regular $C^k$-topology ($\ell + 1 \leq k \leq \infty$).

**Remark.** There is a multi-transversality version of these results, but we do not need it here, see [D4].

Once we have this version of the transversality theorem, we will decompose $\mathcal{H}^3$ into $H$- or IS-orbits $\mathcal{O}_i$ of codimension $\leq n + 1$ and decompose the complement into a finite union of submanifolds $\Gamma_i$ of codimension $> n + 1$. The orbits are exactly those of the generic germs in lists I and II. Using the classification of the preceding section together with the weighted homogeneous decomposition of Section 3, we will show that $\mathcal{H}$ is transverse to the orbits $\mathcal{O}_i$ as a consequence of the next lemma.

**Lemma 5.4.** Let $F(x, t, u) = (\bar{F}(x, t, u), u)$ be a $\mathcal{G}$-versal unfolding of $f(x, t)$, which when viewed as a function of $(x, t)$ is a solution to the heat equation (here $\mathcal{G} = H$ or IS). Then, $\mathcal{H}$ is transverse to $\mathcal{G} \cdot f$ at $f$.

Then, by Theorem 5.1 and the Thom transversality theorem, the set of mappings in either $\mathcal{H}$ or $C^\infty(U, \mathbb{R})$ that are transverse to all of the $\mathcal{O}_i(U)$ and $\Gamma_i(U)$ form a residual set. Then, by a line of reasoning similar to that used by Mather for $\mathcal{J}$-equivalence [MV], we will show that the transversality to the orbits is equivalent to stability in our sense. This will yield Theorem 1.

For Theorem 2, we make use of the following proposition which can be extended to any partial differential equation, see [D4]. Let $U_1 \subset \mathbb{R}^n$ and $0 \in U_2 \subset \mathbb{R}_+$ be open sets.

**Proposition 5.5.** For open $U \subset \mathbb{R}^{n+1}$, $\mathcal{H}$ is a Baire space using the induced Whitney (or regular) $C^k$-topology for $k \geq 2$.

To finish the proof of Theorem 2, we need a simple lemma.

**Lemma 5.6.** For $K \subset \mathbb{R}^n$ compact and $V \subset \mathbb{R}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ a finite dimensional linear subspace of polynomials, suppose $f_0: K \rightarrow \mathbb{R}$ is a $C^k$-map and $\mathcal{U} \subset J^k(K, \mathbb{R})$ is an open neighborhood of $j^k(f_0)(K)$. Then there exists an $\varepsilon > 0$ such that for $p \in B_\varepsilon(0)$, $j^k(f_0 + p)(K) \subset \mathcal{U}$. 
Then we complete the proof of Theorem 2 as follows. We let $U = U_1 \times \text{int}(U_2)$

$$V_r = \left( \sum_{j=1}^{r} W_j \right) \cap \ker(D).$$

We define a map $\Phi: U \times V_{2^n} \to \mathcal{H}^{r}(U)$ by $\Phi(x, p) = j^r(u + p)(x)$, $\ell \geq 3$. By the parametrized transversality theorem applied to the map $\Phi$, the set of $p \in \mathcal{H}^r$ such that the map $j^r(u + p): U \to \mathcal{H}^r(U)$ is transverse to the $C_i(U)$ and $C_i(U)$ forms a dense set because the complement is of measure zero. Also, by Lemma 5.6, given $K \subset U_1$ compact and $\mathcal{U} \subset J^k(K, \mathbb{R})$ an open neighborhood of $j^k(u_0)(K)$, there is an $\varepsilon > 0$ so that for $p \in B_{\varepsilon}(0)$, $j^k(f_0 + p)(K) \subset \mathcal{U}$. Thus, we pick $p \in B_{\varepsilon}(0)$ and belonging to the dense set.

Then, Lemma 5.6 follows by a simple compactness argument applied to the continuous map $\Phi^*: K \times V \to J^k(K, \mathbb{R})$ defined by $\Phi^*(x, p) = j^k(f_0 + p)(x)$. Since $\Phi^*(K \times \{0\}) \subset \mathcal{U}$, there is a ball about 0, $B_{\varepsilon}(0)$ with $\Phi^*(K \times B_{\varepsilon}(0)) \subset \mathcal{U}$.

Thus, the key steps are:

(1) proof of Theorem 5.1 (and Corollary 5.3);
(2) proof that transversality implies stability;
(3) decomposition of $\mathcal{H}^3$ and the proof of Lemma 5.4;
(4) proof of Proposition 5.5.

**Proof of Theorem 5.1 (and Corollary 5.3)**

The theorem will be proven using a standard type argument as, e.g., in [G–G]. We cover $\mathcal{W}^r$ by a countable number of compact sets $\mathcal{W}_i$ and prove

$$\mathcal{S}_\ell = \{ f \in \mathcal{H} : j^r(f) \text{ is transverse to } \mathcal{W}^r \text{ at points of } \mathcal{W}_i \}$$

is open and dense in the regular $C^k$-topology for $k \geq \ell + 1$.

For the openness, we let

$$\mathcal{U}_i = \{ j^r + 1(f) \in \mathcal{H}^{r+1}(U) : j^r(f) \text{ is transverse to } \mathcal{W}^r \text{ at points of } \mathcal{W}_i \}. $$

By an argument similar to that in Proposition 4.5 in [G–G], $\mathcal{U}_i$ is open in $\mathcal{H}^{r+1}(U)$ so $= \mathcal{U}_i \cap \mathcal{H}^{r+1}(U)$ for the open $\mathcal{U}_i = \mathcal{U}_i \cup (J^{r+1}(U, \mathbb{R}) \setminus \mathcal{H}^{r+1}(U))$. Thus, $\mathcal{S}_{\ell} = \{ f \in \mathcal{H} : j^r(f)(U) \subset \mathcal{W}_i \}$ is actually open in the Whitney $C^{r+1}$-topology.

For density, suppose $f \in \mathcal{V} \subset \mathcal{H}$, and $\mathcal{V}$ is an open set in the regular $C^k$-topology defined by the conditions $j^k(f)(K) \subset \mathcal{V}_i$, for $K_i \subset U$ compact, $\mathcal{V}_i$
open in $\mathcal{J}^k(U, \mathbb{R})$, and $1 \leq i \leq r$. We apply the parametrized transversality theorem to the map

$$F: U \times V_{2r} \to \mathcal{J}^r(U, \mathbb{R})$$

$$((x, i), p) \mapsto j^r(f + p)(x, i).$$

For each $(x, i)$ and each $p_0$, the affine map $V_{2r} \to \mathcal{H}^r$ sending $p \mapsto j^r(f + p)(x, i)$ is a submersion at $p_0$, for by Lemma 3.2, given $j^r(h)$ with $h \in \mathcal{H}$, $j^r(h) = j^r(h')$ for $h'$ consisting of the terms of $h$ of weight $\leq 2r$. Thus, $F$ is transverse to $\mathcal{H}^r$, and hence, for almost all $p$, so is $j^r(f + p)$. Then, by applying Lemma 5.6 to each $K_i$ (applied with $m = 0$), we can choose such a $p$ so that $j^r(f + p)(K_i) \subset \mathcal{F}^r_i$, $1 \leq i \leq r$. By the definition of $\mathcal{F}^r$, this implies $f + p \in \mathcal{F}^r$. This proves Theorem 5.1.

For Corollary 5.3, it is only necessary to consider the diagram

$$\begin{array}{ccc}
U & \xrightarrow{j^r(f)} & \mathcal{H}^r(U) \\
\downarrow & & \downarrow \\
\mathcal{W}(U) & \hookrightarrow & \mathcal{W}(U)
\end{array}$$

where $\mathcal{W} = \mathcal{H}^r \cap \mathcal{W}$. Since $\mathcal{H}$ is transverse to $\mathcal{W}$, so is $\mathcal{H}^r(U)$ transverse to $\mathcal{W}(U)$. We conclude that $\mathcal{W}(U)$ is a smooth submanifold and the square in (5.6) is a fiber square. Thus, a simple transversality argument implies that $j^r(f)$ is transverse to $\mathcal{W}(U)$ in $\mathcal{J}^r(U, \mathbb{R})$ iff it is transverse to $\mathcal{W}(U)$ in $\mathcal{H}^r(U)$. Hence, the result is a consequence of Theorem 5.1.

**Proof that Transversality Implies Stability**

The transversality at $(x, i)$ of $j^r(f)$ to $\mathcal{C}^r(U)$ in $\mathcal{J}^r(U, \mathbb{R})$ can be expressed via the equality of tangent spaces

$$d_{(x,i)}(j^r(f))(T_{(x,i)} \mathbb{R}^{n+1}) + T_{j^r(f)(x,i)} \mathcal{C}^r(U) = T_{j^r(f)(x,i)} \mathcal{J}^r(U, \mathbb{R}).$$

(5.7)

To simplify notation we may translate to the origin and delete reference to the point. We follow Mather by expressing (5.7) algebraically. Since $\mathcal{J}^r(U, \mathbb{R}) = U \times \mathcal{J}^r(n + 1, 1) \times \mathbb{R}$ and similarly $\mathcal{C}^r(U) = U \times \mathcal{C}^r \times \mathbb{R}$, (5.7) reduces to

$$d(j^r(f))(T \mathbb{R}^{n+1}) + T_{j^r(f)} \mathcal{C}^r \oplus T \mathbb{R} = T \mathcal{J}^r(n + 1, 1) \oplus T \mathbb{R},$$

(5.8)

where $j^r(f)$ denotes the $\ell$-jet of $f$ at 0 and $j^r(f)$ denotes the jet extension into $\mathcal{J}^r(n + 1, 1)$ by translating all jets to the origin. We can identify algebraically $\mathcal{J}^r(n + 1, 1) \oplus \mathbb{R} \cong \mathcal{C}_{x,i} \oplus \mathbb{R}^{n+1}$ and $T_{j^r(f)} \mathcal{C}^r = T \mathcal{C} \cdot f$, for $\mathcal{C}$ the
appropriate group. Finally, \( d(j'(f))(\mathbb{T} \mathbb{R}^{n+1}) \) is the subspace spanned by \( \{(\partial f/\partial x_1), \ldots, (\partial f/\partial t)\} \). Thus, via (5.8), transversality is equivalent to
\[
\mathbb{T} \mathcal{G} \cdot f + \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial t}, 1 \right) + m'_{x,t}^{x+1} = \mathcal{C}_{x,t}.
\]
(5.9)

Since \( \mathcal{G} \) denotes \( H \) or IS, we obtain that (5.9) is equivalent to
\[
\mathbb{T} \mathcal{G}_e \cdot f + m'_{x,t}^{x+1} = \mathcal{C}_{x,t}.
\]
(5.10)

Since for \( H \) or IS, \( \mathbb{T} \mathcal{G}_e \) (and \( \mathcal{C}_{x,t} \)) is a module over the system of rings \( \mathcal{C}_e \to \mathcal{C}_{x,t} \), for \( H \) or \( \mathbb{R} \to \mathcal{C}_e \to \mathcal{C}_{x,t} \), for IS. Note that \( \mathbb{R} \) is the ring of smooth germs on \( \mathbb{R}^n \) so this is an adequately ordered system of DA-algebras in the sense of [D2]. Hence, by the version of Mather's algebraic lemma for such systems (see [D2b, Lemma 7.4]), there exists an \( \ell \) which depends only on \( n \) and not on the germ \( f \), such that if (5.9) holds then \( \mathbb{T} \mathcal{G}_e \cdot f = \mathcal{C}_{x,t} \), so that \( f \) is \( \mathcal{G} \)-stable. In particular, we will consider orbits \( \mathcal{O} = \mathcal{G} \cdot f \) such that \( \mathbb{T} \mathcal{G} \cdot f = m_{x,t}^{x+1} \), so that if (5.9) holds for \( \ell = 3 \) then it holds for all higher \( \ell \). Conversely, if \( f \) is \( \mathcal{G} \)-stable then (5.9) holds for all \( \ell \) so that \( j'(f) \) is transverse to \( \mathcal{O}'(U) \) for all \( \ell \).

Proof of Lemma 5.4

Since \( \mathcal{G} = H \) or IS, \( 1 \in \mathbb{T} \mathcal{G}_e \cdot f \). First note that if \( d_x f(0) \neq 0 \) then \( \mathbb{T} \mathcal{G} \cdot f = m_{x,t} \) so trivially \( \mathcal{H} \) is transverse to \( \mathcal{G} \cdot f \). Hence, we suppose that \( d_x f(0) = 0 \). By the unfolding theorem, the versality of \( F \) is equivalent to
\[
\mathbb{T} \mathcal{G} \cdot f + \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial t}, 1, \left. \frac{\partial F}{\partial u_1} \right|_{u=0}, \ldots, \left. \frac{\partial F}{\partial u_q} \right|_{u=0} \right) = \mathcal{C}_{x,t},
\]
or
\[
\mathbb{T} \mathcal{G} \cdot f + \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial t} - a_0, \left. \frac{\partial F}{\partial u_1} \right|_{u=0} - a_1, \ldots, \left. \frac{\partial F}{\partial u_q} \right|_{u=0} - a_q \right) = m_{x,t},
\]
(5.11)

where \( a_0 = \partial f/\partial t(0) \) and \( a_i = \partial F/\partial u_i(0) \) for \( i > 0 \). Also,
\[
\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial t}, \left. \frac{\partial F}{\partial u_1} \right|_{u=0}, \ldots, \left. \frac{\partial F}{\partial u_q} \right|_{u=0}
\]
are solutions to the heat equation (as the heat operator commutes with \( \partial/\partial x_1, \partial/\partial t, \) and \( \partial/\partial u_i \) as well as with setting \( u = 0 \)). Thus, (5.11) implies that \( \mathcal{H} \) is transverse to \( \mathcal{G} \cdot f \).
Decomposition of $\mathcal{H}^3$

We will decompose $\mathcal{H}^3$ into $H$- or $IS$-orbits $\mathcal{O}_i$ of codimension $\leq n + 1$ and decompose the complement into a finite union of submanifolds $\mathcal{I}_i$ of codimension $> n + 1$. We claim that the orbits are exactly those of the generic germs in lists I and II. Since those germs are stable they are their own versal unfoldings so by Lemma 5.4, $\mathcal{H}$ is transverse to them. Thus, it remains to show that their complement in $\mathcal{H}^3$ consists of a finite union of submanifolds $\mathcal{I}_i$ of codim $> n + 1$. For this we use the following lemma. By the rank of a germ $f(x, t)$ with $d_x f(0) = 0$, we shall mean the rank of the quadratic form $d_x^2 f(0)$.

**Lemma 5.12.** If $f \in \mathcal{H}$ has rank $> 0$ then $\mathcal{H}$ is transverse to $T \mathcal{G}^3 \cdot f$ ($\mathcal{G} = H$ or $IS$), while those $f \in \mathcal{H}^3$ of rank $= 0$ form a submanifold of codim $= n + (\binom{n}{2})$.

Using this lemma we argue in a standard way as follows. As the action of either $H^3$ or $IS^3$ on $J^3(n + 1, 1)$ is algebraic, the orbits of codimension $> n + 1$ form a closed algebraic subset. As $\mathcal{H}^3$ is a linear subspace, the intersection with it is also a closed algebraic set. It is enough to know that this set has codimension $> n + 1$, for we can then decompose it into a finite union of smooth submanifolds of codimension $> n + 1$. However, by the classification given in Section 4, any germ which is not $H$- (or $IS$-) stable is in the closure of an orbit of either a germ of rank $\leq n - 2$, an $A_2$ germ which is not of type (2) or (3), or an $A_3$ germ. By the lemma, each of these has codimension $> n + 1$; hence, so do their closures.

Finally, for the proof of lemma, we may by an orthogonal change of coordinates assume that the weight 2 terms of $f$ are given by

$$\sum_{i=1}^{n} a_i x_i^2 + 2ct, \quad \text{where} \quad c = \sum_{i=1}^{n} a_i,$$

where at least one $a_i$, say $a_n$, $\neq 0$. Thus,

$$\frac{\partial f}{\partial x_n} \cdot m_{x,i}^k = x_n \cdot m_{x,i}^k \in T \mathcal{G} \cdot f \mod m_{x,i}^{k+2}. \quad (5.13)$$

Since $x_i^3 - 3x_n^2 x_i$, $x_i^2 x_j - x_n^2 x_j$, and $x_i x_j x_k \in H_3$ and $x_i^2 - x_n^2$, $x_i x_j \in H_2$ (recall that $H_k$ denotes the harmonic polynomials of degree $k$), we conclude by (5.13) that

$$m_{x,i}^3 \in T \mathcal{G} \cdot f + (\mathcal{H}^3 \cap m_{x,i}^3) \mod m_{x,i}^4,$$

and

$$m_{x,i}^2 \in T \mathcal{G} \cdot f + (\mathcal{H}^2 \cap m_{x,i}^2) \mod m_{x,i}^3.$$
Likewise by Lemma 3.6, if $\varphi(x)$ denotes a quadratic harmonic polynomial in $x$, and $\ell(x)$ a linear term in $x$, then

\begin{align*}
E(t \cdot \varphi(x)) &\equiv t \cdot \varphi(x), \\
E(t^2 \cdot \ell(x)) &\equiv t^2 \cdot \ell(x), \quad E(t^3) \equiv t^3 \mod m_x^3 + m_{x,t}^4
\end{align*}

and

\begin{align*}
E(t \cdot \ell(x)) &\equiv t \cdot \ell(x), \\
E(t^2) &\equiv t^2 \mod m_x^2 + m_{x,t}^3.
\end{align*}

Thus,

\begin{align*}
m_{x,t}^3 &\in T \cdot \mathcal{G} \cdot f + (\mathcal{H}^3 \cap m_{x,t}^3) \mod m_{x,t}^4, \\
m_{x,t}^2 &\in T \cdot \mathcal{G} \cdot f + (\mathcal{H}^2 \cap m_{x,t}^2) \mod m_{x,t}^3.
\end{align*}

and

\begin{align*}
m_{x,t} &\in \mathcal{H}^1 \mod m_{x,t}^2.
\end{align*}

Hence,

\begin{align*}
m_{x,t} &\in T \cdot \mathcal{G} \cdot f + \mathcal{H} \mod m_{x,t}^4.
\end{align*}

For the last claim we observe that if $f$ has rank 0 then the weight 2 term of $f$ is 0. The codimension of the 0 germ in $W_1 \oplus W_2$ (the weight spaces) is as claimed by Lemma 3.6.

**Proof of Proposition 5.5**

We refer to the proof in Chap. 2 of [G–G] that $C^\infty(X, Y)$, with the Whitney topology, is a Baire space. Given open dense sets $U_i$ and an open set $V$, there is constructed a sequence of functions $f_i \in C^\infty(X, Y)$ such that $f_i \in U_i \cap V$, $f_i$ converges uniformly to $g$ on compact subsets and likewise $j^k(f_i)$ converges uniformly to $j^k(g)$, and finally $g \in V \cap (\cap U_i)$. Now we can repeat the proof choosing $f_i \in \mathcal{W}$. Then, because $k \geq 2$ we have the uniform convergence of the derivatives of $f_i$ order $\leq 2$ to the derivatives of $g$ order $\leq 2$. Since $D(f_i) = 0$, by passing to the limit we find $D(g) = 0$. Hence, $g \in \mathcal{W} \cap V \cap (\cap U_i)$. 

**References**


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