Introduction to Lagrangian and Hamiltonian Mechanics

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Contents

1 Introduction

The goal of this lecture is to provide the basic techniques to tackle problems of classical mechanics to non-physicists. It might also be a good review for physicists after their bachelor before starting with the more advanced classes like advanced quantum mechanics or quantum field theory.
To compile this lecture I used three main sources:
The first one [1] is a script of a Summer School of the German Students Academy, that I attended in the summer of 2002. The course was called "Mathematical Structure of Theoretical Physics" and was held for students still in or fresh out of high school with a special interest in math or physics. The course booklet gives a great overview over the basic and important ideas in almost all areas of theoretical physics.
My second source is the first book of a commonly used German theoretical physics series by Torsten Fliessbach. This is the book I started learning mechanics with and especially for people unfamiliar with the subject it gives a good, but slowly-paced introduction.
The third and final book I based this lecture on, is the first part of an even more famous series - Theoretical Physics by Landau and Lifschitz. These lecture books are ingeniously written, but very hard to follow as a beginner. They are though great books to review the subject after having heard about it in several courses.

2 Review of Newtonian Mechanics

Remark 2.1 *In Mechanics one examines the laws that govern the motion of bodies of matter. Under motion one understands a change of place as a function of time. The motion happens under the influence of forces, that are assumed to be known. [2]*

2.1 Point Mechanics and Newtons First Law

In contrast to continuum mechanics point mechanics deals with bodies that can assumed to be point-like or to be composed of mass points.
A mass point can be completely described, if

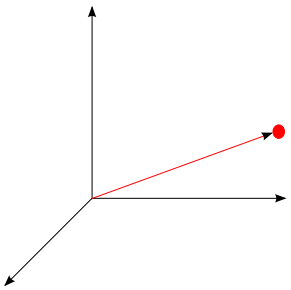

Figure 2.1: The original image

- its mass m, a real number, and
- its position $\vec{r}(t) = (x(t), y(t), z(t))^t$ at a given time, three real numbers in a specified coordinate system,

are known. Motions of such a mass point are then projections of the form $I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$. From this one can define the velocity and acceleration:

Definition 2.1 The velocity is given by

$$\dot{\vec{r}}(t) = \frac{d\vec{r}}{dt}, \tag{2.1}$$

while the acceleration is defined by

$$\ddot{\vec{r}}(t) = \frac{d\vec{r}^2}{dt^2} = \frac{d\dot{\vec{r}}}{dt}. \tag{2.2}$$

In this context time can be seen as a independent process and therefore a free parameter. The position $\vec{r}(t)$ is however a dynamic variable, which means that is development in time is given by the equations of motions (e.q.m.)

$$m\ddot{\vec{r}}(t) = \vec{F}(\vec{r}(t), \dot{\vec{r}}(t), \ldots, t). \tag{2.3}$$

This is also Newtons first Axiom.

The basic challenge of Newtonian Mechanics is to find the solution to eq. 2.3 for a given force $\vec{F}$ with the boundary conditions $\vec{r}(t_0), \dot{\vec{r}}(t_0)$.

2.2 Potential and Kinetic Energy

Especially simple forces are of the form

$$\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r}). \tag{2.4}$$

These are called potential forces, where $V(\vec{r})$ stands for the potential energy.
From this one can already derive the form of the kinetic energy. Taking eq. 2.3 and inserting $\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ we get

$$m\ddot{r}_i(t) = -\frac{d}{dr_i}V(r_i). \tag{2.5}$$

Taking the scalar product with $\dot{r}_i(t)$ this becomes

$$m\ddot{r}_i(t)\dot{r}_i(t) = -\frac{dr_i}{dt}\frac{d}{dr_i}V(r_i). \tag{2.6}$$

Using the chain rule and therefore the conditions $\frac{d}{dt}(m\dot{r}_i(t)^2) = 2m\dot{r}_i\ddot{r}_i$ and $\frac{d}{dt}V(r_i(t) = \sum_i \frac{dV_i}{dr_i}\frac{dr_i}{dt}$ we can rewrite our equation as a total differential:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{\vec{r}}(t)^2 + V(\vec{r}(t))\right) = 0 \tag{2.7}$$

This means we have a quantity that is constant over time. We recognize this to be our total energy $E(t)$ and from this we can conclude that the kinetic energy is of the form

$$T(t) = \frac{1}{2}m\dot{\vec{r}}(t)^2. \tag{2.8}$$

3 Principle of Least Action

Remark 3.1 *The most general formulation of the laws governing the motion of mechanical systems is the "Principle of Least Action" or "Hamilton's Principle", according to which every mechanical system is characterised by a definite function*

$$\mathscr{L}(r_1, r_2, \ldots, r_s, \dot{r_1}, \dot{r_2}, \ldots, \dot{r_s}, t) = \mathscr{L}(r, \dot{r}, t) \tag{3.1}$$

and the motion of the system is such, that a certain condition is satisfied [3].

3.1 Derivation of the Lagrange Equations

The condition that needs to be satisfied is the following:
Let the mechanical system fulfill the boundary conditions $r(t_1) = r^{(1)}$ and $r(t_2) = r^{(2)}$. Then the condition on the system is that it moves between these positions in such a way that the integral

$$S = \int_{t_1}^{t_2} \mathscr{L}(r, \dot{r}, t)dt \tag{3.2}$$

is minimized. Here S is called the action (hence also the name of the theorem) and $\mathscr{L}$ is the Lagrangian of the system given.
To find a general solution to eq. 3.2, one can apply the tools of variational calculus. Assuming, for simplicity, that the system has only one degree of freedom , let $r = r(t)$ be the function for which S is a minimum. This means that S is going to increase, when $r(t)$ is replaced by any function of the form $r(t) + \delta r(t)$, where $\delta r(t)$ is called a small variation. Since our boundary conditions ($r(t_1) = r^{(1)}$ and $r(t_2) = r^{(2)}$) should still hold, it directly follows that $\delta r(t_1) = \delta r(t_2) = 0$. Applying this change $r \longrightarrow r + \delta r$ to S, we get

$$\tilde{S} = \int_{t_1}^{t_2} \mathscr{L}(r + \delta r, \dot{r} + \dot{\delta r}, t)dt - \int_{t_1}^{t_2} \mathscr{L}(r, \dot{r}, t)dt \tag{3.3}$$

If we expand this to first order we get

$$\tilde{S} = \int_{t_1}^{t_2} \left(\mathscr{L}(r, \dot{r}, t) + \frac{\partial \mathscr{L}}{\partial r}\delta r + \frac{\partial \mathscr{L}}{\partial \dot{r}}\delta \dot{r} \right) dt, \tag{3.4}$$

which leads to

$$\delta S = \delta \int_{t_1}^{t_2} \mathscr{L}(r, \dot{r}, t)dt = \int_{t_1}^{t_2} \left(\frac{\partial \mathscr{L}}{\partial r}\delta r + \frac{\partial \mathscr{L}}{\partial \dot{r}}\delta \dot{r} \right) dt = 0. \tag{3.5}$$

Since $\delta\dot{r} = \frac{d\delta r}{dt}$ we can by integration of parts rewrite this as

$$\delta S = \left[\frac{\partial \mathscr{L}}{\partial \dot{r}}\delta r\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial \mathscr{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{r}}\right)\delta r dt = 0 \tag{3.6}$$

Remembering our boundary conditions for δr, we now have an integral that must vanish for all values of δr. This can only be the case if the integrand is identically zero and therefore we get

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{r}}\right) - \frac{\partial \mathscr{L}}{\partial r} = 0 \tag{3.7}$$

This can easily be generalized to a system with more degrees of freedom. Then one can obtain s equations of the form

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{r}_i}\right) - \frac{\partial \mathscr{L}}{\partial r_i} = 0, \tag{3.8}$$

where $i = 1, 2, \ldots, s$.
These sets of differential equations for a given system are called the "Lagrange's Equations" and are the equations of motion of the system (they give the relations between acceleration, velocities and coordinates). They constitute a set of of s second-order differential equations. The general solution therefore contains $2s$ constants, which can be determined from the initial conditions of the system (e.g. initial values of coordinates and velocities).

3.2 Properties of the Lagrange Function

3.2.1 Additivity

If a mechanical system consists of two separate parts A and B, each part would have a separate Lagrangian $\mathscr{L}_A$ and $\mathscr{L}_B$ respectively. If these parts do not interact, e.g. in the limit where the distance between the parts become so large that the interaction can be neglected, the Lagrangian of the system is given by $\mathscr{L} = \mathscr{L}_A + \mathscr{L}_B$. This additivity states that the equations of motion of part A can not be dependent on quantities pertaining to part B and vice versa.

3.2.2 Uniqueness

If one considers two functions $\mathscr{L}'(r, \dot{r}, t)$ and $\mathscr{L}(r, \dot{r}, t)$ differing only by a total derivative w.r.t. time of some function $f(r, t)$

$$\mathscr{L}'(r, \dot{r}, t) = \mathscr{L}(r, \dot{r}, t) + \frac{d}{dt}f(r, t) \tag{3.9}$$

one finds that the action

$$S' = \int_{t_1}^{t_2} \mathscr{L}'(r, \dot{r}, t)dt = \int_{t_1}^{t_2} \mathscr{L}(r, \dot{r}, t)dt + \int_{t_1}^{t_2} f(r, t)dt = S + f(r^{(2)}, t_2) - f(r^{(1)}, t_1) \tag{3.10}$$

differs only by a quantity which gives zero on variation. This means the conditions $\delta S = 0$ and $\delta S' = 0$ are equivalent and the form of the equations of motion is unchanged. Therefore the Lagrangian is only defined up to an additive total time derivative of any function of coordinates and time.

4 Lagrangian Mechanics

4.1 Galileo's Relativity Principle

To write down the equations of motion for a certain problem, one first has to choose a frame of reference. The goal is then to find a frame of reference in which the laws of mechanics take their simplest form.
If the physical world would be "arbitrary", one could end up choosing a frame of reference in which space and time could be inhomogeneous and anisotropic (which would lead to BIG problems...). But luckily it can be derived that one can always find a frame of reference in which space is homogeneous and isotropic and time is homogeneous. This is then called an "inertial frame". In such a frame a free body which is at rest at some instant of time always remains at rest.
From this knowledge about the homogeneity one can deduct properties of the Lagrange function $\mathscr{L}$. This implies that $\mathscr{L}$ can not depend explicitly on the vector $\vec{r}$ of the particle or the time t. Since space is isotropic, it can also not depend on the direction of $\vec{v}$ and must therefore only be function of $\vec{v}^2$.
Since the Lagrangian is independent of $\vec{r}$, we have $\frac{\partial \mathscr{L}}{\partial \vec{r}} = 0$, and so Lagrange's equation becomes

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\vec{r}}}\right) = 0. \tag{4.1}$$

From this we see that $\frac{\partial \mathscr{L}}{\partial \dot{\vec{r}}} = const.$ and since it is a function of velocity only, it follows that

$$\vec{v} = const. \tag{4.2}$$

From this we can conclude that in an inertial frame any free motion takes place with a velocity which is constant in both magnitude and direction. This is the "Law of Inertia" or "Newton's First Law". More importantly we see from this that if we consider another frame moving uniformly in a straight line relative to the inertial frame, then the laws of free motion will be the same in both frames!

Remark 4.1 *Experiment shows that not only are the laws of free motion the same in the two frames, but the frames are entirely equivalent in all mechanical aspects. Thus there is not one, but an infinity of inertial frames moving, relative to another, uniformly in a straight line. In all these frames the properties of space and time are the same and so are the laws of mechanics. This constitutes "Galileo's Relativity Principle". [3]*

4.2 The Lagrangian of a Free Particle

The next logical step is to derive the form of the Lagrangian for a given system. Let us begin with the simplest example, the free motion of a particle relative to an inertial frame of reference. As we have already seen above the Lagrangian in this case can only depend on the velocity. In order to get to the real form of dependence we use "Galileo's Relativity Principle".
If two frames K and K' move with an infinitesimal velocity $\vec{\epsilon}$ to each other, then $\vec{v'} = \vec{v} + \vec{\epsilon}$. But since the equations of motion must have the same form in every frame, the Lagrangians $\mathscr{L}(v^2)$ and $\mathscr{L}'(v^2)$ can only differ by a total time derivative of a function of coordinates and time (see sect. 3.2.2). This way we can write

$$\mathscr{L}' = \mathscr{L}(\vec{v}'^2) = \mathscr{L}(\vec{v}^2 + 2\vec{v}\cdot\vec{\epsilon} + \vec{\epsilon}^2) \approx \mathscr{L}(\vec{v}^2) + \frac{\partial \mathscr{L}}{\partial v^2} 2\vec{v}\cdot\vec{\epsilon} \tag{4.3}$$

The term $\frac{\partial \mathscr{L}}{\partial v^2} 2\vec{v}\cdot\vec{\epsilon}$ can only be a total time derivative if it is a linear function of the velocity $\vec{v}$. Therefore $\frac{\partial \mathscr{L}}{\partial v^2}$ is independent of the velocity, which means that the Lagrangian can be written as

$$\mathscr{L} = \frac{1}{2} m v^2. \tag{4.4}$$

The quantity m which appears in the Lagrangian is the mass of the particle. Remembering the additive property of the Lagrangians we can show that for a system of particles, which do not interact, the Lagrangian is given as

$$\mathscr{L} = \sum \frac{1}{2} m_a v_a^2. \tag{4.5}$$

Remark 4.2 *The above definition of mass becomes only meaningful, when we take the additive property into account. A Lagrangian can always be multiplied by an arbitrary constant without affecting the equations of motions; such multiplications then amount to a change in the unit of mass. The ratios of the masses remain unchanged by this and it is only these ratios which are physically meaningful. [3]*

4.3 The Lagrangian for a System of Particles

Instead of one free particle on can also look at a system of particles which interact with each other. This new system can be described by taking the Lagrangian for the non-interacting particles (eq. 4.5) and adding a certain function of the coordinates, depending on the nature of the interaction:

$$\mathscr{L} = \sum \frac{1}{2} m_a v_a^2 - U(\vec{r_1}, \vec{r_2}, \ldots) \tag{4.6}$$

The first part of the Lagrangian we can recognize to be T, the kinetic energy, and the second part to be U, the potential energy.

Furthermore we can see the homogeneity and isotropy of time, since t can be replaced with $-t$ without changing the Lagrangian and the equations of motion. Knowing the Lagrangian above, we can now also derive the equations of motion for this system. They are given by

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \vec{v_a}} = \frac{\mathscr{L}}{\partial \vec{r_a}} \tag{4.7}$$

Inserting $\mathscr{L}$ in this we get

$$m_a \frac{d\vec{v}_a}{dt} = -\frac{\partial U}{\partial \vec{r}_a} \tag{4.8}$$

These are "Newton's Equations", the basis of classical mechanics.
The vector $\vec{F} = -\frac{\partial U}{\partial \vec{r}_a}$ is called force. Like U, it depends only on the coordinates of the particles, not their velocities. Furthermore one can see that the potential Energy, U, is only defined up to a constant, which has no effect on the equations of motion.

4.4 Generalised Coordinates and External Fields

Inside our inertial frames our choice of coordinates is arbitrary, so we could also transform the coordinates in general like this

$$x_a = f_a(r_1, r_2, \ldots, r_s) \text{ and } \dot{x_a} = \sum_k \frac{\partial f_a}{\partial r_k}\dot{r}_k, \tag{4.9}$$

which would lead to a Lagrangian of the form

$$\mathscr{L} = \frac{1}{2}\sum_{i,k} a_{ik}(r)\dot{r}_i\dot{r}_k - U(r). \tag{4.10}$$

In this case the kinetic energy is still a quadratic function of the velocities, but might also depend on the coordinates.

The motion of a system in an external field is described by the same type of Lagrangian, the only difference that can appear is the explicit dependence of the kinetic energy on time. For example, if we have a particle that moves in an external field the Lagrangian is given by $\mathscr{L} = \frac{1}{2}m\dot{v}^2 - U(\vec{r}, t)$.

Remark 4.3 *It is often necessary to deal with mechanical systems in which the interactions between different bodies take the form of constraints. These constraints can be very complicated and hard to incorporate, when working directly with the differential equations of motion. Here it is often easier to construct a Lagrange function out of the kinetic and potential energy of the system and derive the correct equations of motion from there.*

5 Symmetries

As mentioned before when writing down the equations of motions we end up with a system of differential equations that has $2s$ degrees of freedom. In a closed mechanical system the number of degrees of freedom is reduced to $2s-1$, because time can be chosen up to an additive constant since nothing depends directly on it. Now there exist functions of these $2s-1$ quantities whose values remain constant during the motion and depend only on the initial condition. Such functions are called integrals of motion. But not all integrals of motion are of equal importance in mechanics. There are some of profound importance, which can be derived from the fundamental properties of space and time (homogeneity and isotropy).These quantities are then said to be conserved.

5.1 Homogeneity of Time

By virtue of this homogeneity the Lagrangian of a closed system does not explicitly depend on time (see above). The total time derivative of the Lagrangian can therefore be written as

$$\frac{d\mathscr{L}}{dt} = \sum_i \frac{\partial \mathscr{L}}{\partial r_i}\dot{r}_i + \frac{\partial \mathscr{L}}{\partial \dot{r}_i}\ddot{r}_i \tag{5.1}$$

(If the Lagrangian would depend explicitly on time, a term $\frac{\partial \mathscr{L}}{\partial t}$ would need to be added on the right-hand side.) Applying Lagrange's equation to that we get

$$\frac{d\mathscr{L}}{dt} = \sum_i \frac{d}{dt}\left(r_i \frac{\partial \mathscr{L}}{\partial \dot{r}_i}\right) \tag{5.2}$$

or

$$\frac{d}{dt}\left(\sum_i r_i \frac{\partial \mathscr{L}}{\partial \dot{r}_i} - \mathscr{L}\right) = 0. \tag{5.3}$$

This shows us that the quantity $E \equiv \sum_i r_i \frac{\partial \mathscr{L}}{\partial \dot{r}_i} - \mathscr{L}$ is conserved. It is the energy of the system. In a closed system $\mathscr{L}$ is given as $T-U$. Since we can also write $\sum_i r_i \frac{\partial \mathscr{L}}{\partial \dot{r}_i} = 2T$ (using Euler's theorem on homogeneous functions), we have now re-derived our statement of the beginning that $E = 2T - \mathscr{L} = T + U$.

5.2 Homogeneity and Isotropy of Space

A second conservation law follows from the homogeneity of space. By virtue of this the mechanical properties of a closed system are unchanged by any parallel

displacement of the entire system in space (e.g. a pendulum will swing the same way here or 2m displaced to the right). This leads to the conservation of Momentum. A third conserved quantity can be derived out of the isotropy of space. This means that the mechanical properties of a system are not changed, when the system is rotated as a whole in any manner in space. From this the conservation of angular momentum can be shown.

5.3 Symmetries

There are many conserved quantities as the ones mentioned above. In fact there is a conserved quantity for every invariance toward a one-parametric symmetry transformation (Noethertheorem) [2].
Often ones spends a lot of time to derive the Lagrangian of a system just to find in this way the conserved quantities. This is for example a common check in theoretical particle physics to derive the known conserved quantities when trying to devise new models.

6 Hamiltonian Mechanics

The formulation of classical mechanics in terms of the Lagrangian is based on formulating the state of a mechanical system in terms of coordinates ($r(t)$) and velocities ($\dot{r}(t)$). But this is not the only possibility! Sometimes it is more advantageous to choose another set of quantities to describe a mechanical system, e.g. generalised coordinates and momenta.

6.1 Derivation of the Hamilton Equations

The transformation of one set of variables, coordinates and velocities, to another, generalised coordinates and momenta, can be carried out via a Legendre transformation. Let us try to transform our Lagrange function into a quantity that is described by generalised coordinates and momenta.
The total differential of the Lagrangian is given by

$$d\mathscr{L} = \sum_i \frac{\partial \mathscr{L}}{\partial r_i} dr_i + \sum_i \frac{\partial \mathscr{L}}{\partial \dot{r}_i} d\dot{r}_i = \sum_i \dot{p}_i dr_i + \sum_i p_i d\dot{r}_i, \tag{6.1}$$

since $\frac{\partial \mathscr{L}}{\partial \dot{r}_i}$ are by definition the generalised momenta and $\frac{\partial \mathscr{L}}{\partial r_i} = \dot{p}_i$ by the Lagrange equation. Now we can rewrite the second term using the chain rule $\sum_i p_i d\dot{r}_i = d\left(p_i \dot{r}_i\right) - \sum_i \dot{r}_i dp_i$ to

$$d\mathscr{L} = \sum_i \dot{p}_i dr_i + d\left(p_i \dot{r}_i\right) - \sum_i \dot{r}_i dp_i. \tag{6.2}$$

Rearranging this we end up with

$$d\left(p_i \dot{r}_i - \mathscr{L}\right) = -\sum_i \dot{p}_i dr_i + \sum_i \dot{r}_i dp_i. \tag{6.3}$$

The argument of the total differential on the left side is (as we have seen before when talking about symmetries) the energy of the system, now only expressed in terms of coordinates and momenta. In this form we call it the "Hamiltonian" or "Hamilton's function" of the system

$$H(p, r, t) = p_i \dot{r}_i - \mathscr{L} \tag{6.4}$$

From the second part of eq.6.3 we get to know that the total differential of the Hamiltonian is given by

$$dH = -\sum_i \dot{p}_i dr_i + \sum_i \dot{r}_i dp_i. \tag{6.5}$$

Using this we can derive the required equations of motion in terms of p and r:

$$\dot{r}_i = \frac{\partial H}{\partial p_i} \; and \; \dot{p}_i = -\frac{\partial H}{\partial r_i} \tag{6.6}$$

These are the so-called "Hamilton's equations". They form a set of $2s$ first-order differential equations for the $2s$ unknown functions $p_i(t)$ and $r_i(t)$. Because of their simplicity and symmetry they are also called "canonical equations".

6.2 Uniqueness

The total time derivative of the Hamiltonian is given by

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial r_i}\dot{r}_i + \frac{\partial H}{\partial p_i}\dot{p}_i, \tag{6.7}$$

which with the use of eq.6.6 reduces to

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{6.8}$$

In particular it holds that if the Hamiltonian does not explicitly depend on time, then $\frac{dH}{dt} = 0$, and energy is conserved.

Remark 6.1 *The difference between the Lagrangian and Hamiltonian formalism is only a Legendre transformation. For some problems it is easier to write $\mathscr{L}$ down, for others H. One is given by a system of s second-order differential equations, while the other one describes the same physical problem with $2s$ first-order differential equations. The Hamiltonian is in general though the more popular tool to use, since it has a closer relation to the above mentioned symmetries (Noethertheorem).*

Bibliography

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[3] L.D. Landau and E.M. Lifschitz. *Mechanics*. Butterworth, Oxford, 1981.