AN EQUIVARIANT TIETZE EXTENSION THEOREM FOR PROPER ACTIONS OF LOCALLY COMPACT GROUPS

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ABSTRACT. The classical Tietze extension theorem asserts that any continuous map $f: A \rightarrow \mathbb{R}^n$ from a closed subset A of a normal space X admits a continuous extension $F: X \rightarrow \mathbb{R}^n$. The Tietze-Gleason theorem is an equivariant version of the same theorem for spaces with actions of compact groups, proved by A. Gleason in the 1950s. Here we prove the following version of the theorem for proper actions of locally compact groups: let G be a locally compact group acting properly on a completely normal space X such that X/G is paracompact, and let A be a closed G -invariant subset of X . Suppose $\rho: G \rightarrow GL(n, \mathbb{R})$ is a representation; now any continuous G -equivariant map $f: A \rightarrow \mathbb{R}^n(\rho)$ admits a continuous G -equivariant extension $F: X \rightarrow \mathbb{R}^n(\rho)$.

1. INTRODUCTION

The Tietze extension theorem is one of the most basic, and perhaps the most well-known, continuous extension theorems. An equivariant version of it for compact groups was proven by A. Gleason in the 1950s [G], using the Haar integral to "average" over the group. The same technique does not work for non-compact groups since the Haar integral does not generally converge over non-compact groups. However, for proper actions of locally compact groups, we can use slices to pass from a solution for a compact subgroup on a slice, to a solution for the whole group on the corresponding tubular neighborhood.

2. PREREQUISITES

Let G be a locally compact Hausdorff topological group, and let X be a topological space. An *action* of G on X is a continuous map $\Phi: G \times X \rightarrow X$ such that $\Phi(e, x) = x$ for all $x \in X$, where e is the neutral element of the group, and such that $\Phi(g', \Phi(g, x)) = \Phi(g'g, x)$. We usually denote $\Phi(g, x) = gx$.

Two subsets A and B of a topological space X are said to be *separated* in case $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. A topological space X is *completely normal* if any two separated subsets A and B of X can be separated by disjoint neighborhoods V_A and V_B . A topological space X is completely normal if and only if any subspace of X is normal [?, Theorem II.5.1].

The action of G on a completely regular space X is a *Cartan action* if, for any $x \in X$, there exists a neighborhood V of x in X such that $\{g \in G \mid gV \cap V \neq \emptyset\}$ is relatively compact in G . The action is *proper* if for any pair of points $x, y \in X$ we can find neighborhoods V_x and V_y of x and y respectively, such that the set $\{g \in G \mid gV_x \cap V_y \neq \emptyset\}$ is relatively compact in G . Even more restrictively, we call the action *Palais proper* if for each $x \in X$ we can find a neighborhood V_x of x in X such that for any other point $y \in X$ we can find a neighborhood V_y of y in X such that the set $\{g \in G \mid gV_x \cap V_y \neq \emptyset\}$ is relatively compact in G .

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It is clear that any Palais proper action is proper, and that any proper action is a Cartan action.

Let H be a closed subgroup of G . A subset S of a G -space X is an H -slice if GS is open in X and there exists a G -map $f: GS \rightarrow G/H$ such that $S = f^{-1}(eH)$.

Lemma 1. *Suppose that $S \subset X$ is an H -slice. Then*

- i) $gS \cap S \neq \emptyset$ implies that $g \in H$,
- ii) S is closed.

Proof. i) Suppose that $gS \cap S \neq \emptyset$, and let $f: GS \rightarrow G/H$ be such that $S = f^{-1}(eH)$. Then for some $s \in S$, also $gs \in S$ and thus $gH = g(eH) = gf(s) = f(gs) \in f(S) = eH$. But that implies $g \in H$.
 ii) This is trivial since f is continuous and eH is closed (it is a point in G/H , which is Hausdorff because H is compact). □

Given a closed subgroup H of G and an H -space S there is an action of H on the product $G \times S$ given by $h(g, s) = (gh^{-1}, hs)$. We denote by $G \times_H S$ the quotient space $(G \times S)/H$, which is called the *twisted product* of G and S with respect to H . There is an action of G on $G \times_H S$ defined by the formula $\bar{g}[g, s] = [\bar{g}g, s]$.

Proposition 2. *Let H be a closed subgroup of G and let S be an H -invariant subset of X . Then S is an H -slice if and only if GS is an open subset of X and*

$$G \times_H S \approx_G GS.$$

We say that the open set GS is a *tubular neighborhood* (of x) if S is an H -slice (a slice at x). A *tubular covering* of a G -space X is a covering of X by tubular neighborhoods.

3. THE EQUIVARIANT TIETZE EXTENSION THEOREM

Theorem 3. *Let G be a locally compact group and let X be a completely normal proper G -space such that X/G is paracompact. Let $A \subset X$ be a closed G -subset. Let $\rho: G \rightarrow \text{GL}(n, \mathbb{R})$ be a representation and let $\varphi: A \rightarrow \mathbb{R}^n(\rho)$ be a G -map. Then there exists a continuous G -extension $\psi: X \rightarrow \mathbb{R}^n(\rho)$.*

By [B, Theorem 2.5], any Cartan G -space X has a tubular covering $\{GS_i\}_{i \in I}$ where each S_i is an H_i -slice for some compact subgroup H_i of G .

Lemma 4. *Let G be a locally compact group, let X and Y be G -spaces, let S be a H -slice for a compact subgroup H of G , and let V be an H -invariant subset of S . Let $f: V \rightarrow Y$ be an H -map. Then there exists a unique continuous G -map $\tilde{f}: GV \rightarrow Y$ which extends f .*

Proof. The only possible G -function $GV \rightarrow Y$ which extends f is $\tilde{f}$ defined by

$$\tilde{f}(gv) = gf(v) \quad \forall g \in G, v \in V.$$

We show that the function $\tilde{f}$ is well-defined. If $gv = g'v'$ for some $g, g' \in G$ and $v, v' \in V \subset S$ then $v = g^{-1}g'v'$ implying $S \cap (g^{-1}g')S \neq \emptyset$. Since S is a slice, we get by Lemma 1 that $g^{-1}g' \in H$. Since f is an H -map, then $f(v) = f(g^{-1}g'v') = g^{-1}g'f(v')$ giving $gf(v) = g'f(v')$. Hence $\tilde{f}(gv) = \tilde{f}(g'v')$ and $\tilde{f}$ is well defined.

Equivariance is clear from the definition, and $\tilde{f}$ is continuous since the diagram below commutes and the left vertical projection is a quotient map because it is the projection $G \times V \rightarrow (G \times V)/H$ to the orbit space of a compact group action.

$$\begin{array}{ccccc}
 (g, v) & & G \times V & \xrightarrow{\text{id} \times f} & G \times Y \\
 \downarrow & & \downarrow \text{quotient} & & \downarrow \text{action} \\
 gv & & GV & \xrightarrow{\tilde{f}} & Y
 \end{array}$$

Corollary 5. *Let $i \in I$, and denote by $A_i = A \cap GS_i$. The map $f_i = f|_{A_i} : A_i \rightarrow \mathbb{R}^n(\rho)$ admits a G -equivariant continuous extension $F_i : GS_i \rightarrow \mathbb{R}^n(\rho)$.*

Proof. The map

$$\tilde{f}_i : A_i \cap S_i \rightarrow \mathbb{R}^n(\rho|)$$

where $\rho| : H_i \rightarrow GL(n, \mathbb{R})$, admits an H_i -equivariant extension

$$\tilde{F}_i : S_i \rightarrow \mathbb{R}^n(\rho|)$$

by [G]. By Lemma 4 there exists a G -equivariant continuous extension

$$F_i : GS_i \rightarrow \mathbb{R}^n(\rho)$$

given by $F_i(gs) = g\tilde{F}_i(s)$, and we easily see that $F_i|_{A_i} = f_i$. $\square$

Lemma 6. *Suppose that $\{U_i\}_{i \in I}$ is a locally finite covering of a topological space X . Then $\{\bar{U}_i\}_{i \in I}$ is locally finite as well.*

Proof. Suppose $x \in X$; then there exists a neighborhood U of x which meets only finitely many U_i . But if $U \cap U_j = \emptyset$, then $U \subset U_j^C$ and then since U is open, $U \subset \text{int}(U_j^C) = (\bar{U}_j)^C$, so $U \cap \bar{U}_j = \emptyset$. Hence U meets $\bar{U}_i$ only for those $i \in I$ for which U meets U_i , and there are only finitely many such $i \in I$. $\square$

In particular, the covering $\{\overline{GS_i}\}_{i \in I}$ is locally finite.

Since X/G is paracompact, the covering $\{GS_i\}_{i \in I}$ has a locally finite open G -refinement $\{U'_j\}_{j \in J}$. Because Hausdorff paracompact spaces are normal and $\{U'_j\}_{j \in J}$ is, in particular, point-finite, we may find a G -refinement $\{U_j\}_{j \in J}$ of $\{U'_j\}_{j \in J}$ such that $\bar{U}_j \subset U'_j$ for all $j \in J$. ($\bar{U}_j = \text{cl}_X U_j$.) By Lemma 6, there exists a continuous G -equivariant extension $F_i : GS_i \rightarrow \mathbb{R}^n(\rho)$ of f_i .

Well-order the index set I , and let $i \in I$. Suppose that for all $j < i$ we have extensions $F_j : A_j \rightarrow Y$ of $f|_{A \cap \bar{U}_j}$, where $A_j = A \cup \bar{V}_j$ and where $\bar{V}_j$ is the closure in $\bar{U}_j$ of a neighborhood V_j of $A \cap \bar{U}_j$ in $\bar{U}_j$, such that all F_j agree on their common area of definition. We claim that there exists a neighborhood V_i of $A \cap \bar{U}_i$ in $\bar{U}_i$ and an extension $F_i : A_i \rightarrow Y$ of $f|_{A \cap \bar{U}_i}$ where $A_i = A \cup \bar{V}_i$ ($\bar{V}_i = \text{cl}_{\bar{U}_i} V_i$), which agrees with every F_j , $j < i$, on their common area of definition.

The functions F_j where $j < i$ combine to a function $\bar{F}_i : \bigcup_{j < i} A_j \rightarrow Y$. This function $\bar{F}_i$ is continuous by the local finiteness of the family $\{\bar{V}_j\}_{j < i} \cup \{A\}$ (since $\bigcup_{j < i} A_j = A \cup \bigcup_{j < i} \bar{V}_j$). Furthermore, $\bigcup_{j < i} A_j$ is closed, again by the local finiteness of the family $\{\bar{V}_j\}_{j < i} \cup \{A\}$.

Now by Lemma 6 the map $\bar{F}_i|_{(\bar{U}_i \cap (\bigcup_{j < i} A_j))}$ has a G -extension $\tilde{F} : \bar{V}_i \rightarrow Y$ where V_i is a neighborhood of $\bar{U}_i \cap (\bigcup_{j < i} A_j)$ in $\bar{U}_i$. In particular V_i is a neighborhood of $A \cap \bar{U}_i$ in $\bar{U}_i$. Now we use this to obtain a map $F_i : A_i \rightarrow Y$ where $A_i = A \cup \bar{V}_i$, by setting:

$$\begin{aligned}
 F_i|_A &= \bar{F}_i|_A = f|_A \\
 F_i|_{\bar{V}_i} &= \tilde{F}
 \end{aligned}$$

Obviously F_i agrees with each F_j ($j < i$) on their common area of definition.

Hence by transfinite construction there exists, for every $i \in I$, a neighborhood V_i of $A \cap \overline{U}_i$ in $\overline{U}_i$ and an extension $F_i: A_i \rightarrow Y$ of $f|_{A \cap \overline{U}_i}$ where $A_i = A \cup \overline{V}_i$ such that F_i agrees with any F_j on their common area of definition.

Using the above we obtain a map $F: \overline{W} \rightarrow Y$ where $W = \bigcup_{i \in I} V_i \cap U_i$ by setting

$$F(x) = F_i(x) \text{ when } x \in \overline{V}_i.$$

The map is well-defined; it is continuous by local finiteness and closedness of the $\overline{V}_i$, and W is a neighborhood of A since

$$A \subset \bigcup_{i \in I} (A \cap U_i) \subset \bigcup_{i \in I} (V_i \cap U_i) = W.$$

□

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