A topological manifold is homotopy equivalent to some CW-complex

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Chapter 1

Introduction

1.1 Thanks

First of all, I would like to thank my supervisor Erik Elfving for suggesting the topic and for giving valuable feedback while I was writing the thesis.

1.2 The problem

The goal of this Pro Gradu thesis is to show that a topological manifold has the same homotopy type as some CW complex. This will be shown in several "parts":

A) A metrizable ANR has the same homotopy type as some CW complex.

   i) For any ANR Y there exists a dominating space X of Y which is a CW complex.

   ii) A space which is dominated by a CW complex is homotopy equivalent to a CW complex.

B) A topological manifold is an ANR.

1.3 Notation and terminology

Just a few remarks on notation: By a mapping (map) I will always mean a continuous single-valued function.

By a neighborhood of a point x or a subset A of a topological space X I will always mean an open subset of X containing the point x or the set A unless otherwise is stated.

A covering, however, does not have to be made up by open sets. If it is, then I will refer to it as an open covering. Similarly, a closed covering is a covering which consists only of closed sets.

I will assume that anything which can be found in Väisälä's Topologia I-II is already familiar.

Some notation:

$I = [0, 1] \subset \mathbb{R}$
$\mathbb{Z}^+ = \mathbb{N} = \{1, 2, 3, ...\}$
$\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$
$\mathbb{R}_+ = [0, \infty[$
$\dot{\cup}$ = disjoint union.

1.4 Continuity of combined maps

This section contains a couple of useful basic lemmas which will be used many times throughout the thesis.

Reference: [7]

Suppose that $\{X_i : i \in I\}$ is a family of subspaces of a topological space X such that $X = \bigcup_{i \in I} X_i$, and suppose that Y is some topological space. Assume that for each $i \in I$ there is defined a mapping $f_i : X_i \to Y$ such that if $X_i \cap X_j \neq \emptyset$ then $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$. We wish to define a new combined mapping $f : X \to Y$ by setting $f|X_i = f_i$ for all $i \in I$, and the question is whether such a function would be continuous or not.

Lemma 1.4.1 (The glueing lemma). *Assume that I is finite and that each X_i is a closed subset of X. Then f is continuous.*

Proof. Let A be a closed subset of Y - then $f^{-1}(A) = \cup_{i \in I} f_i^{-1}(A)$ is closed since each $f_i^{-1}(A)$ is closed in X_i and thus in X (X_i is closed in X) by the continuity of f_i and the union is finite. Hence f is continuous. □

Lemma 1.4.2. *If x is an interior point of one of the X_i, then f is continuous in x.*

Proof. Note that there is now no restriction on the set I, and the X_i are not necessarily closed. Let x be an interior point of, say, X_1 and let U be a neighborhood of $f(x)$ in Y. Since f_1 is continuous there is a neighborhood

V of x in X_1 such that $f(V) = f_1(V) \subset U$. Now V is open in X_1 and so $V = X_1 \cap W$ for some open subset W of X, and hence $V' = V \cap Int(X_1) = W \cap Int(X_1)$ is an open neighborhood of x in X and $f(V') \subset f(V) \subset U$. Hence f is continuous in x. □

Definition 1.4.3 (Neighborhood-finiteness (also called local finiteness)). *A family $\{A_\alpha : \alpha \in \mathscr{A}\}$ of sets in a topological space X is called neighborhood-finite if each point in X has a neighborhood V such that $V \cap A_\alpha \neq \emptyset$ for only finitely many $\alpha \in \mathscr{A}$.*

Lemma 1.4.4. *If $\{X_i : i \in I\}$ is a neighborhood-finite closed covering of X, then f is continuous.*

Proof. Let $x \in X$ be arbitrarily chosen; it now suffices to show that f is continuous in x. Since $\{X_i : i \in I\}$ is neighborhood-finite, there exists a neighborhood U of x which meets only finitely many X_i. Now $U \cap X_i$ is closed in U for all i and so by Lemma (1.4.1) the restriction $f|_U$ is continuous. Now we may add U to the original collection of X_is; it no longer satisfies the assumptions of this lemma but since x is an interior point of U, f is continuous in x by Lemma (1.4.2) Now f is continuous in all of X since x was arbitrarily chosen.

□

1.5 Paracompact spaces

The goal of this chapter is to prove that a metrizable space is paracompact. Reference: [1]

Proposition 1.5.1. *Let $\{A_\alpha : \alpha \in \mathscr{A}\}$ be a neighborhood-finite family in a topological space X. Then:*

(A) $\{\overline{A}_\alpha : \alpha \in \mathscr{A}\}$ is also neighborhood-finite.

(B) For each $\mathscr{B} \subset \mathscr{A}$, $\bigcup\{\overline{A}_\beta : \beta \in \mathscr{B}\}$ is closed in X.

Proof. (A) Let $x \in X$. Then there is a neighborhood $U(x)$ such that $A_\alpha \cap U(x) = \emptyset$ for all except finitely many α. If $A_\alpha \cap U(x) = \emptyset$ for some α, then $A_\alpha \subset U(x)^c$, and since $U(x)$ is open it follows that $\overline{A}_\alpha \subset U(x)^c$ and so $\overline{A}_\alpha \cap U(x) = \emptyset$ and so (1) holds.

(B) Let $B = \bigcup_{\beta\in\mathscr{B}} \overline{A}_\beta$. Now, if $x \notin B$, then by (A) there is a neighborhood U of x which meets at most finitely many $\overline{A}_\beta$, say $\overline{A}_{\beta_1}, ..., \overline{A}_{\beta_n}$. In that case, $U \cap \bigcap_{i=1}^n \overline{A}_{\beta_i}^c$ is a neighborhood of x not meeting B and hence B^c is open.

□

Proposition 1.5.2. *Let $\{E_\alpha : \alpha \in \mathscr{A}\}$ be a family of sets in a topological space Y, and let $\{B_\beta : \beta \in \mathscr{B}\}$ be a neighborhood-finite* closed *covering of Y. Assume that each B_β interesects at most finitely many sets E_α. Then each E_α can be embedded in an open set $U(E_\alpha)$ such that the family $\{U(E_\alpha) : \alpha \in \mathscr{A}\}$ is neighborhood-finite.*

Proof. For each α define $U(E_\alpha) = Y - \bigcup\{B_\beta : B_\beta \cap E_\alpha = \emptyset\}$. Each $U(E_\alpha)$ is open by 1.5.1 (B), since $\{B_\beta\}$ is a neighborhood-finite family of closed sets. We show that $\{U(E_\alpha) : \alpha \in \mathscr{A}\}$ is neighborhood-finite:

It follows from the definition of $U(E_\alpha)$ that $B_\beta \cap U(E_\alpha) \neq \emptyset \Leftrightarrow B_\beta \cap E_\alpha \neq \emptyset$. Therefore, since each B_{β_i} intersects at most finitely many E_α, the set B_{β_i} intersects at most finitely many $U(E_\alpha)$. By the neighborhood-finiteness of $\{B_\beta\}$ any $y \in Y$ has a neighborhood V intersecting only finitely many $B_{\beta_i}, i = 1, ..., n$, and hence $V \subset \bigcup_{i=1}^{n} B_{\beta_i}$ which as a finite union intersects only finitely many $U(E_\alpha)$. Since $E_\alpha \subset U(E_\alpha)$ for all α then the claim holds. □

Definition 1.5.3 (Refinement of a covering). *A refinement of a covering $\{A_\alpha : \alpha \in \mathscr{A}\}$ of a topological space $X$is a covering $\{B_\beta : \beta \in \mathscr{B}\}$ such that for every set B_β where $\beta \in \mathscr{B}$ there exists a set A_α where $\alpha \in \mathscr{A}$ such that $B_\beta \subset A_\alpha$.*

Example 1.5.4. *A subcovering is a refinement of the original covering.*

Definition 1.5.5 (Paracompact space). *A Hausdorff space Y is* paracompact *if every open covering of Y has an open neighborhood-finite refinement.*

Example 1.5.6. *A discrete space is paracompact.*
A compact space is paracompact.

Theorem 1.5.7 (E. Michael). *Let Y be a regular space. The following are equivalent:*

(A) Y is paracompact.

(B) Each open covering of Y has an open refinement that can be decomposed into an at most countable collection of neighborhood-finite families of open sets (not necessarily coverings).

(C) Each open covering of Y has a neighborhood-finite refinement, whose sets are not necessarily open or closed.

(D) Each open covering of Y has a closed neighborhood-finite refinement.

Proof. "(A) $\Rightarrow$ (B)"
Follows from the definition of paracompactness.

"(B) $\Rightarrow$ (C)"
Let $\{U_\beta : \beta \in \mathscr{B}\}$ be an open covering of Y. By (B) there is an open refinement $\{V_\gamma : \gamma \in \mathscr{G}\}$ where $\mathscr{G} = \bigcup_{n\in\mathbb{N}} \mathscr{A}_n$ is a disjoint union such that$\{V_\alpha : \alpha \in \mathscr{A}_n\}$ is a neighborhood-finite family of open sets (but not necessarily a covering).

For each $n \in \mathbb{N}$, let $W_n = \bigcup_{\alpha\in\mathscr{A}_n} V_\alpha$. Now $\{W_n : n \in \mathbb{N}\}$ is an open covering of Y. Define $A_i = W_i - \bigcup_{j<i} W_j$. Then $\{A_i : i \in \mathbb{N}\}$ is a covering, since $\bigcup_{i\in\mathbb{N}} A_i = \bigcup_{i\in\mathbb{N}} W_i = Y$ and so $\{A_i\}$ is a refinement of $\{W_i\}$. Furthermore, $\{A_i\}$ is neighborhood-finite, since the neighborhood $W_{n(y)}$ of $y \in Y$, where $n(y)$ is the first $i \in \mathbb{N}$ for which $y \in W_i$, does not intersect A_i whenever $i > n(y)$.

Claim: Now $\{A_n \cap V_\alpha : \alpha \in \mathscr{A}_n, n \in \mathbb{N}\}$ *is a refinement of* $\{U_\beta\}$.
Proof: Let $y \in Y$. Then there exists $n \in \mathbb{N}$ and $\alpha \in \mathscr{A}_n$ such that $y \in V_\alpha$. Let n_0 be the smallest such integer n. Then $y \in V_{\alpha_0}$ for some $\alpha_0 \in \mathscr{A}_{n_0}$, and $y \in W_{n_0}$ but $y \notin \cup_{i<n_0} W_i$; hence $y \in A_{n_0}$ and thus $y \in A_{n_0} \cap V_{\alpha_0}$. Thus $\{A_n \cap V_\alpha : \alpha \in \mathscr{A}_n, n \in \mathbb{N}\}$ is a covering, and clearly it is a refinement. □

Moreover it is neighborhood-finite since each $y \in Y$ has a neighborhood intersecting at most finitely many A_n, and for each n the point y has a neighborhood intersecting at most finitely many V_α where $\alpha \in \mathscr{A}_n$.

"(C) $\Rightarrow$ (D)" Let $\mathscr{A}$ be an open covering. To each $y \in Y$, associate a neighborhood $U_y \in \mathscr{A}$ of y. Now, since Y is regular, there exists disjoint neighborhoods of y and U_y^c - let V_y be the neighborhood of y. It follows that $y \in V_y \subset \overline{V}_y \subset U_y$. The family $\{V_y : y \in Y\}$ is an open covering of Y; hence, by the assumption it has a neighborhood-finite refinement $\{A_y : y \in Y\}$. By Proposition (1.5.1) $\{\overline{A}_y : y \in Y\}$ is also neighborhood-finite, and $\overline{A}_y \subset \overline{V}_y \subset U_y$; hence $\{\overline{A}_y : y \in Y\}$ is a closed neighborhood-finite refinement of $\mathscr{A}$. Hence every open covering of Y has a closed neighborhood-finite refinement.

"(D) $\Rightarrow$ (A)" Let $\mathscr{U}$ be an open covering of Y, and let $\mathscr{E}$ be its closed neighborhood-finite refinement. Now for each $y \in Y$ there exists a neighborhood V_y which meets at most finitely many sets $E \in \mathscr{E}$. Using $\{V_y : y \in Y\}$, find a closed neighborhood-finite refinement $\mathscr{B}$. Since each $B \in \mathscr{B}$ intersects at most finitely many sets $E \in \mathscr{E}$, then by Proposition (1.5.2) each $E \in \mathscr{E}$ can be embedded into an open set $G(E)$, such that $\{G(E) : E \in \mathscr{E}\}$ is neighborhood-finite. If we associate to each $E \in \mathscr{E}$ a set $U(E) \in \mathscr{U}$ such that $E \subset U(E)$, then $\{U(E) \cap G(E)\}$ is a neighborhood-finite open refinement of

$\mathscr{U}$.

$\square$

Definition 1.5.8 (Star, barycentric refinement, star refinement). *Let $\mathfrak{U}$ be a covering of a space Y. For any $B \subset Y$, the set $\bigcup\{U_\alpha \in \mathfrak{U} : B \cap U_\alpha \neq \emptyset\}$ is called the* star *of B with respect to $\mathfrak{U}$, denoted $St(B, \mathfrak{U})$.*

A covering $\mathfrak{B}$ is called a barycentric refinement *of the covering $\mathfrak{U}$ if the covering $\{St(y, \mathfrak{B}) : y \in Y\}$ refines $\mathfrak{U}$.*

A covering $\mathfrak{B} = \{V_\beta : \beta \in \mathscr{B}\}$ is a star refinement *of the covering $\mathfrak{U}$ if the covering $\{St(V_\beta, \mathfrak{B}) : \beta \in \mathscr{B}\}$ refines $\mathfrak{U}$.*

Note that if $\mathfrak{B}$ is a star refinement of $\mathfrak{U}$ then it is also a barycentric refinement, since for each $y \in Y$ there exists a set $V \in \mathfrak{B}$ such that $y \in V$, and clearly $St(y, \mathfrak{B}) \subset St(V, \mathfrak{B}) \subset U$ for some $U \in \mathfrak{U}$.

If every covering of a space Y has a barycentric refinement, then it also has a star refinement:

Proposition 1.5.9. *Let $\mathfrak{U}$ be a covering of a space Y. A barycentric refinement $\mathfrak{D}$ of a barycentric refinement $\mathfrak{B}$ of $\mathfrak{U}$ is a star refinement of $\mathfrak{U}$.*

Proof. Let $W_0 \in \mathfrak{D}$, and choose some $y_0 \in W_0$. For each $W \in \mathfrak{D}$ such that $W \cap W_0 \neq \emptyset$, choose a $z \in W \cap W_0$. Then $W \cup W_0 \subset St(z, \mathfrak{D}) \subset V$ for some $V \in \mathfrak{B}$. Because, then, $y_0 \in V$ it follows that $V \subset St(y_0, \mathfrak{B})$ and so $St(W_0, \mathfrak{D}) \subset St(y_0, \mathfrak{B}) \subset U$ for some $U \in \mathfrak{U}$, since $\mathfrak{B}$ is a barycentric refinement of $\mathfrak{U}$. $\square$

Thus a covering $\mathfrak{U}$ has a star refinement if and only if it has a barycentric refinement.

Theorem 1.5.10 (Stone). *A T_1 space Y is paracompact if each open covering has an open barycentric refinement.*

Proof. Let $\mathfrak{U} = \{U_\alpha : \alpha \in \mathscr{A}\}$ be an open covering of Y. We will show that it has a refinement as required in Theorem (1.5.7) (B).

Let $\mathfrak{U}^*$ be an open star refinement of $\mathfrak{U}$ (exists by Proposition (1.5.9)), and let $\{\mathfrak{U}_n : n \geq 0\}$ be a sequence of open coverings such that each $\mathfrak{U}_{n+1}$ star refines $\mathfrak{U}_n$ when $n \geq 0$, and $\mathfrak{U}_0$ star refines $\mathfrak{U}^*$.

Define a new sequence of coverings:

$$\begin{aligned}
&\mathfrak{B}_1 = \mathfrak{U}_1\\
&\mathfrak{B}_2 = \{St(V, \mathfrak{U}_2) : V \in \mathfrak{B}_1 (= \mathfrak{U}_1)\}\\
&\vdots\\
&\mathfrak{B}_n = \{St(V, \mathfrak{U}_n) : V \in \mathfrak{B}_{n-1}\}
\end{aligned}$$

$\vdots$

Claim i): Each covering $\{St(V,\mathfrak{U}_n) : V \in \mathfrak{B}_n\}$ *refines* $\mathfrak{U}_0$.
Proof: $\mathbf{n=1}$ By definition, since $\mathfrak{B}_1 = \mathfrak{U}_1$ and $\mathfrak{U}_1$ star refines $\mathfrak{U}_0$.

$\mathbf{n>1}$ Assume that the claim holds for $n = k-1$. Let $V \in \mathfrak{B}_k \Rightarrow V = St(V_0, \mathfrak{U}_k)$ for some $V_0 \in \mathfrak{B}_{k-1}$. Denote by $\{V_i : i \in I\}$ the set of neighborhoods $V_i \in \mathfrak{U}_k$ such that $V_i \cap V_0 \neq \emptyset$. Then

$$\begin{aligned} St(V,\mathfrak{U}_k) &= St[St(V_0,\mathfrak{U}_k),\mathfrak{U}_k] \\ &= (\bigcup_{j\in J} V_j) \quad \text{where} \quad j \in J \Leftrightarrow V_j \in \mathfrak{U}_k \quad \text{and} \quad V_j \cap V_i \neq \emptyset \quad \text{for some} \quad i \in I. \end{aligned}$$

If $V_j \cap V_i \neq \emptyset$ for some $i \in I$ then , because $\mathfrak{U}_k$ star refines $\mathfrak{U}_{k-1}$, there exists $V' \in \mathfrak{U}_{k-1}$ such that

$$V_j \cup V_i \subset St(V_i, \mathfrak{U}_k) \subset V'$$

and since $V_i \cap V_0 \neq \emptyset$ then

$$V_0 \cap V' \supset V_0 \cap (V_j \cup V_i) = (V_0 \cap V_j) \cup (V_0 \cap V_i) \neq \emptyset,$$

hence $V' \subset St(V_0, \mathfrak{U}_{k-1})$, and so $V_j \subset V' \subset St(V_0, \mathfrak{U}_{k-1})$.
Thus we have shown that $St(V,\mathfrak{U}_k) = St[St(V_0,\mathfrak{U}_k),\mathfrak{U}_k] \subset St(V_0,\mathfrak{U}_{k-1})$.

From the induction assumption it then follows that $St(V,\mathfrak{U}_k) \subset St(V_0,\mathfrak{U}_{k-1}) \subset U$ for some $U \in \mathfrak{U}_0$, and so $\{St(V,\mathfrak{U}_n) : V \in \mathfrak{B}_n\}$ refines $\mathfrak{U}_0$. $\square$

Claim ii): Each $\mathfrak{B}_n$ *is an open refinement of* $\mathfrak{U}_0$.
Proof: Since the $\mathfrak{U}_i$ are open coverings, the $\mathfrak{B}_n$ are trivially open coverings.

$\mathbf{n=1}$ By definition $\mathfrak{B}_1 = \mathfrak{U}_1$ star refines $\mathfrak{U}_0$, so $\mathfrak{B}_1$ is an open refinement of $\mathfrak{U}_0$.

$\mathbf{n>1}$ Assume that $\mathfrak{B}_{n-1}$ is an open refinement of $\mathfrak{U}_0$ and that $V \in \mathfrak{B}_{n-1}$. Then

$$St(V,\mathfrak{U}_n) = \bigcup_{i\in I} U_i$$

where $i \in I \Leftrightarrow U_i \in \mathfrak{U}_n$ and $, U_i \cap V \neq \emptyset$. Since $\mathfrak{U}_n$ refines $\mathfrak{U}_{n-1}$ then each $U_i \subset V_i \in \mathfrak{U}_{n-1}$ and $St(V,\mathfrak{U}_n) \subset St(V,\mathfrak{U}_{n-1})$, and by the previous claim, $St(V,\mathfrak{U}_{n-1}) \subset U$ for some $U \in \mathfrak{U}_0$ and so $\mathfrak{B}_n = \{St(V,\mathfrak{U}_n) : V \in \mathfrak{B}_{n-1}\}$ refines $\mathfrak{U}_0$. $\square$

Now well-order Y and for each $(n, y) \in \mathbb{N} \times Y$ define

$$E_n(y) = St(y, \mathfrak{B}_n) - \bigcup_{z<y}\{St(z, \mathfrak{B}_{n+1})\}.$$

Claim iii): $\mathfrak{F} = \{E_n(y) : (n, y) \in \mathbb{Z}^+ \times Y\}$ is a covering of Y, and $\mathfrak{F}$ refines $\mathfrak{U}^*$.
Proof: Given $p \in Y$, the set

$$A = \{z \in Y : p \in \bigcup_{i=1}^{\infty} St(z, \mathfrak{B}_i)\}$$

is nonempty, since $p \in A$. If y is the first member of A, then $p \in St(y, \mathfrak{B}_n)$ for some $n \in \mathbb{N}$ and $p \notin St(z, \mathfrak{B}_{n+1})$ for all $z < y$. Hence $p \in E_n(y)$, and thus $\mathfrak{F}$ is a covering.

Furthermore, if $V \in \mathfrak{F}$ then $V = E_n(y)$ for some $(n, y) \in \mathbb{N} \times Y$ and thus $V \subset St(y, \mathfrak{B}_n) \subset St(y, \mathfrak{U}_0)$ since $\mathfrak{B}_n$ refines $\mathfrak{U}_0$ and $St(y, \mathfrak{U}_0) \subset St(U, \mathfrak{U}_0) \subset W$ where $y \in U \in \mathfrak{U}_0$ and $W \in \mathfrak{U}^*$, since $\mathfrak{U}_0$ star refines $\mathfrak{U}^*$. Hence $\mathfrak{F}$ refines $\mathfrak{U}^*$. $\Box$

Claim iv): Each $U \in \mathfrak{U}_{n+1}$ can meet at most one $E_n(y)$.
Proof: If$U \in \mathfrak{U}_{n+1}$ is such that $U \cap E_n(y) \neq \emptyset$ then there exists a set $V \in \mathfrak{B}_n$ such that $y \in V$ and $U \cap V \neq \emptyset$, and therefore $y \in V \cup U \subset St(V, \mathfrak{U}_{n+1}) \in \mathfrak{B}_{n+1}$. It follows that $U \subset St(V, \mathfrak{U}_{n+1}) \subset St(y, \mathfrak{B}_{n+1})$. Hence, if U meets $E_n(y)$ then it cannot meet $E_n(p)$ for $p > y$. $\Box$

Denote $W_n(y) = St(E_n(y), \mathfrak{U}_{n+2})$.

Claim v): $\mathfrak{W} = \{W_n(y) : (n, y) \in \mathbb{N} \times Y\}$ is an open covering of Y.
Proof: Let $p \in Y$. Now by Claim iii) there exists $(n, y) \in \mathbb{Z}^+ \times Y$ such that $p \in E_n(y)$. Since $\mathfrak{U}_{n+2}$ is a covering there exists a set $U \in \mathfrak{U}_{n+2}$ such that $p \in U$ and hence $U \cap E_n(y) \neq \emptyset$ which gives $U \subset St(E_n(y), \mathfrak{U}_{n+2})$ and hence $p \in U \subset W_n(y)$. Moreover, $\mathfrak{W}$ is open since $\mathfrak{U}_{n+2}$ is open. $\Box$

Claim vi): $\mathfrak{W}$ refines $\mathfrak{U}$.
Proof: If $V \in \mathfrak{W}$ then $V = St(E_n(y), \mathfrak{U}_{n+2})$ for some $(n, y) \in \mathbb{Z}^+ \times Y$. Since by Claim iii), $\mathfrak{F}$ refines $\mathfrak{U}^*$ we have $St(E_n(y), \mathfrak{U}_{n+2}) \subset St(V, \mathfrak{U}_{n+2})$ for some $V \in \mathfrak{U}^*$. Furthermore, since $\mathfrak{U}_{n+2}$ refines $\mathfrak{U}^*$ we have $St(V, \mathfrak{U}_{n+2}) \subset St(V, \mathfrak{U}^*) \subset U$ for some $U \in \mathfrak{U}$ since $\mathfrak{U}^*$ star refines $\mathfrak{U}$. $\Box$

Claim vii): The family $\mathfrak{W}_n = \{W_n(y) : y \in Y\}$ is neighborhood-finite for fixed $n \in \mathbb{N}$.

Proof: Let $U \in \mathfrak{U}_{n+2}$. Since

$$\begin{aligned} U \cap W_n(y) \neq \emptyset &\Leftrightarrow U \cap St(E_n(y), \mathfrak{U}_{n+2}) \neq \emptyset \\ &\Leftrightarrow \exists V \in \mathfrak{U}_{n+2} s.t. V \cap U \neq \emptyset \quad \text{and} \quad V \cap E_n(y) \neq \emptyset \\ &\Leftrightarrow E_n(y) \cap St(U, \mathfrak{U}_{n+2}) \neq \emptyset \end{aligned}$$

and because $St(U, \mathfrak{U}_{n+2}) \subset U_0 \in \mathfrak{U}_{n+1}$ where U_0 meets at most one $E_n(y)$, it follows that U can meet at most one $W_n(y)$. $\square$

Hence, since $\mathfrak{W} = \bigcup_{n\in\mathbb{N}} \mathfrak{W}_n$, we have proved that the covering $\mathfrak{W}$ satisfies the conditions in (B) in 1.5.7, and it remains to show that the space Y is regular.

Claim viii): The space Y is regular.
Proof: Let B be a closed subset of Y and let $y \in Y - B$. Since Y is T_1, $\{y\}$ is closed in Y. Hence $\mathfrak{U} = \{Y - y, B^c\}$ is an open covering of Y. Let $\mathfrak{B}$ be an open star refinement of $\mathfrak{U}$. Then $St(y, \mathfrak{B})$ and $St(B, \mathfrak{B})$ are disjoint neighborhoods of y and B:

Assume that there are neighborhoods V and V' in $\mathfrak{B}$ such that $y \in V$, $B \cap V' \neq \emptyset$ and $V \cap V' \neq \emptyset$. Then $y \in St(V, \mathfrak{B})$ and $V' \subset St(V, \mathfrak{B})$ and thus $St(V, \mathfrak{B}) \subsetneq Y - y$ and $St(V, \mathfrak{B}) \subsetneq B^c$; hence $\mathfrak{B}$ is not a star refinement of $\mathfrak{U}$, which is a contradiction.

Hence Y is regular, and it follows from 1.5.7 (B) that Y is paracompact. $\square$

Definition 1.5.11 (Locally starring sequence). *Let $\mathfrak{U} = \{U_\alpha : \alpha \in \mathscr{A}\}$ be an open covering of Y. A sequence $\{\mathfrak{U}_n : n \in \mathbb{N}\}$ of open coverings is called* locally starring *for $\mathfrak{U}$ if for each $y \in Y$ there exists a neighborhood V of y and an $n \in \mathbb{N}$ such that $St(V, \mathfrak{U}_n) \subset U_\alpha$ for some $\alpha \in \mathscr{A}$.*

Theorem 1.5.12 (Arhangel'skii). *A T_1 space is paracompact if for each open covering $\mathfrak{U}$ there exists a sequence $\{\mathfrak{U}_n : n \in \mathbb{N}\}$ of open coverings that is locally starring for $\mathfrak{U}$.*

Proof. Let $\mathfrak{U} = \{U_\alpha : \alpha \in \mathscr{A}\}$ be a covering of Y and $\{\mathfrak{U}_n : n \in \mathbb{N}\}$ a sequence of open coverings that is locally starring for $\mathfrak{U}$. We can assume that $\mathfrak{U}_{n+1}$ refines $\mathfrak{U}_n$ for all $n \in \mathbb{N}$. (If not, replace $\mathfrak{U}_{n+1}$ with $\{U_j \cap U_i : U_i \in \mathfrak{U}_n, U_j \in \mathfrak{U}_{n+1}\}$.) Let

$$\mathfrak{B} = \{V \text{ open in } Y | \exists\, n : [V \subset U \in \mathfrak{U}_n] \wedge [St(V, \mathfrak{U}_n) \subset U_\alpha \text{ for some } \alpha \in \mathscr{A}]\}.$$

For each $V \in \mathfrak{B}$, let $n(V)$ be the smallest integer satisfying the condition.

Claim: $\mathfrak{B}$ is an open covering of Y.
Proof: $\mathfrak{B}$ consists of open sets by definition. If $y \in Y$ then since $\{\mathfrak{U}_n\}$ is locally starring for $\mathfrak{U}$ there exists a neighborhood $V(y)$ of y such that $St(V(y), \mathfrak{U}_n) \subset U_\alpha$ for some $n \in \mathbb{N}, \alpha \in \mathscr{A}$. Since $\mathfrak{U}_n$ is a covering of Y there exists a set $U \in \mathfrak{U}_n$ such that $y \in U$. Let $W = V(y) \cap U \neq \emptyset$. Then $St(W, \mathfrak{U}_n) \subset St(V(y), \mathfrak{U}_n) \subset U_\alpha$ where $\alpha \in \mathscr{A}$ and $W \subset U \in \mathfrak{U}_n$. The set W is open as the intersection of two open sets; hence $y \in W \in \mathfrak{B}$, and hence $\mathfrak{B}$ is a covering of Y. $\square$

Claim: The covering $\mathfrak{B}$ is a barycentric refinement of $\mathfrak{U}$.
Proof: For some $y \in Y$, let $n(y) = \min\{n(V) : (y \in V) \wedge (V \in \mathfrak{B})\}$, and let $V_0 \in \mathfrak{B}$ be such that $y \in V_0$ and $n(V_0) = n(y)$. For any $V \in \mathfrak{B}$ where $y \in V$ we have $n(V) \geq n(y)$, so

$$St(y, \mathfrak{B}) \subset \bigcup\{St(y, \mathfrak{U}_i) : i \geq n(y)\}.$$

Since $\mathfrak{U}_{i+1}$ refines $\mathfrak{U}_i$ it follows that $St(y, \mathfrak{B}) \subset St(y, \mathfrak{U}_{n(y)}) = St(y, \mathfrak{U}_{n(V_0)}) \subset U_\alpha$ for some $\alpha \in \mathscr{A}$. Hence $\mathfrak{B}$ is a barycentric refinement for $\mathfrak{U}$ $\square$.

It follows from Theorem (1.5.10) that Y is paracompact.

$\square$

Theorem 1.5.13 (Stone). *A metrizable space is paracompact.*

Proof. We will prove the theorem by finding a sequence of open coverings which is locally starring for all open coverings of the metrizable space X, and using 1.5.12.

Let d be a metric for the space X and denote

$$\mathfrak{B}_n = \{B(x, \frac{1}{n}) : x \in X\} \quad \forall \quad n \in \mathbb{N}.$$

Given an open covering $\{U_\alpha : \alpha \in \mathscr{A}\}$ and a point $x \in X$, choose an $n \in \mathbb{N}$ such that $d(x, U_\alpha^c) \geq \frac{1}{n} > 0$. By letting $V(x) = B(x, \frac{1}{3n})$, then $St(V(x), \mathfrak{B}_{3n}) \subset U_\alpha$. (If $y \in St(V(x), \mathfrak{B}_{3n})$ and $z \in V(x)$ then $d(z, y) < \frac{2}{3n}$ and so

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{1}{3n} + \frac{2}{3n} = \frac{1}{n} \leq d(x, U_\alpha^c)$$

and hence $y \in U_\alpha$.)

Thus $\{\mathfrak{B}_n\}$ is locally starring for any open covering of X. By Theorem (1.5.12), X is paracompact.

$\square$

1.6 Properties of normal and fully normal spaces

Reference: [1], [2], [6]

This section contains some useful properties of normal spaces plus the definition of and some lemmas concerning fully normal spaces, which will come in handy later.

A covering $\{V_\lambda : \lambda \in \Lambda\}$ is *point-finite* if for each point $y \in Y$ there are at most finitely many indices $\lambda \in \Lambda$ such that $y \in V_\lambda$. An interesting result is that normal spaces are characterized by the "shrinkability" of open point-finite coverings:

Lemma 1.6.1. *Let X be a T_1 topological space. Then the following properties are equivalent:*

a) X is normal.

b) Let $\alpha = \{V_\lambda : \lambda \in \Lambda\}$ be a point-finite covering of a normal space X, then α has an open refinement $\beta = \{U_\lambda : \lambda \in \Lambda\}$ such that $\overline{U}_\lambda \subset V_\lambda$ for each $\lambda \in \Lambda$, and $U_\lambda \neq \emptyset$ whenever $V_\lambda \neq \emptyset$.

Proof. "(a) $\Rightarrow$ (b)"
Well-order the indexing set Λ and for each $x \in X$, denote

$$h(x) = \max\{\lambda : x \in V_\lambda\}.$$

Now, $h(x)$ is well defined since x is only contained in finitely many V_λ.

Well-order $\mathscr{P}(X)$ - we will define a map $\phi : \Lambda \to \mathscr{P}(X)$ by transfinite construction such that $U_\lambda = \phi(\lambda)$ is an open set for all λ and

i) $\overline{U}_\lambda \subset V_\lambda, U_\lambda \neq \emptyset$ whenever $V_\lambda \neq \emptyset$.

ii) $\{U_\alpha : \alpha \leq \lambda\} \cup \{V_\beta : \beta > \lambda\}$ is a covering of X for all $\lambda \in \Lambda$.

Assume that $\phi(\alpha)$ is defined for all $\alpha < \lambda$, and note that then

$$\{U_\alpha : \alpha < \lambda\} \cup \{V_\beta : \beta \geq \lambda\}$$

is a covering of X.

It follows that

$$F = X \setminus [\bigcup_{\alpha<\lambda} U_\alpha \cup \bigcup_{\beta>\lambda} V_\beta] \subset V_\lambda$$

and, since F is the complement of an open set it is closed and hence by the normality of X there is an open set U such that $F \subset U \subset \overline{U} \subset V_\lambda$ (If $F = \emptyset$ then replace F with a point in U_λ). Let $\phi(\lambda) = U_\lambda$ be the first such set in the well-ordering of $\mathscr{P}(X)$. Then, clearly, the conditions i) and ii) are satisfied by the new family.

Hence we have a uniquely defined family of sets U_λ such that $\overline{U}_\lambda \subset V_\lambda \ \forall\ \lambda \in \Lambda$. It remains to show that $\{U_\lambda : \lambda \in \Lambda\}$ is a covering of X.

Assume that $x \in X$ is an arbitrary point; then $x \notin \bigcup_{\beta>h(x)} V_\beta$ and hence by the condition ii) $x \in U_\alpha$ for some $\alpha \leq h(x)$.

"(b) $\Rightarrow$ (a)"

Let A and B be disjoint closed sets in X. Then $\{A^c, B^c\}$ is a point-finite covering of X, and so there is an open refinement $\{U_1, U_2\}$ such that $\overline{U}_1 \subset A^c$ and $\overline{U}_2 \subset B^c$. Then $\overline{U}_1^c$ is a neighborhood of A, $\overline{U}_2^c$ is a neighborhood of B, and

$$\overline{U}_1^c \cap \overline{U}_2^c = (\overline{U}_1 \cup \overline{U}_2)^c = X^c = \emptyset$$

and hence X is normal.

□

Definition 1.6.2 (Fully normal space). *A Hausdorff space X is* fully normal *if every open covering has an open barycentric refinement (see Definition (1.5.8)).*

Proposition 1.6.3. *A fully normal space is normal.*

Proof. Let A and B be disjoint closed subsets of X - now $\{A^c, B^c\}$ is an open covering of X.

Let $\mathfrak{U} = \{U_j : j \in J\}$ be an open barycentric refinement of $\{A^c, B^c\}$. Define

$$V_A = \bigcup \{U_j : j \in J \text{ and } A \cap U_j \neq \emptyset\}$$

$$V_B = \bigcup \{U_j : j \in J \text{ and } B \cap U_j \neq \emptyset\},$$

now V_A and V_B are open neighborhoods of A and B, and we will see that they are disjoint:

Suppose that $x \in U_{j_A} \cap U_{j_B}$, where $U_{j_A} \cap A \neq \emptyset$ and $U_{j_B} \cap B \neq \emptyset$. Then $St(x, \mathfrak{U}) \nsubseteq A^c$ and $St(x, \mathfrak{U}) \nsubseteq B^c$ and so $\mathfrak{U}$ is not a *barycentric* refinement, and we have a contradiction.

$\square$

Theorem 1.6.4. *A metrizable space is fully normal.*

Proof. Let X be a metrizable space, and let $\mathfrak{U} = \{U_i : i \in I\}$ be an open covering of X. Since X is metrizable it is paracompact, and hence $\mathfrak{U}$ has a neighborhood-finite open refinement $\mathfrak{V} = \{V_j : j \in J\}$. A neighborhood-finite covering is certainly point-finite, and so by Lemma (1.6.1), since a metrizable space is normal, $\mathfrak{V}$ has an open refinement $\mathfrak{W} = \{W_j : j \in J\}$ such that $\overline{W}_j \subset V_j$ for all $j \in J$.

Now each $x \in X$ has a neighborhood U_x which intersects only finitely many V_j. Denote by $J(x)$ the set of indices $j \in J$ such that $x \in \overline{W}_j$ and let $K(x)$ be the set of indices $k \in J$ for which U_x intersects V_k but $x \notin \overline{W}_k$. Then both $J(x)$ and $K(x)$ are finite.

Denote

$$B_x = U_x \cap \bigcap_{j \in J(x)} V_j \cap \bigcap_{k \in K(x)} \overline{W}_k^c.$$

$\mathfrak{B} = \{B_x : x \in X\}$ is an open cover of X since the B_x are finite intersections of open sets containing x, and it is actually a barycentric refinement of $\mathfrak{U}$:

Let $x \in X$; now there is a W_j which contains x, since $\mathfrak{W}$ is a covering of X. If $x \in B_y$ then $\overline{W}_j$ intersects B_y and so $j \notin K(y)$ by the definition of B_y. Since $x \in B_y \cap W_j$ we have $U_y \cap V_j \neq \emptyset$ and so $j \in J(y)$ since $j \notin K(y)$ and so $B_y \subset V_j$. Hence $St(x, \mathfrak{B}) \subset V_j \subset U_i$ for some $i \in I$ and so $\mathfrak{B}$ is a barycentric refinement of $\mathfrak{U}$.

$\square$

Chapter 2

Retracts

2.1 Extensors and Retracts

This section contains the basic definitions and properties of the spaces called absolute extensors/retracts (AE/AR) and absolute neighborhood extensors/retracts (ANE/ANR). In a later chapter we will see that in metrizable spaces the concepts of AE and AR (or ANE and ANR) are essentially the same.

Definition 2.1.1 (Weakly hereditary topological class of spaces). *A* weakly hereditary topological class of spaces *(WHT) is a class $\mathscr{C}$ of spaces satisfying the following conditions:*

(WHT 1) $\mathscr{C}$ is topological*: If $\mathscr{C}$ contains a space X then it contains every homeomorphic image of X.*

(WHT 2) $\mathscr{C}$ is weakly hereditary*: If $\mathscr{C}$ contains a space X then it contains every closed subspace of X.*

Example 2.1.2. *The following classes of spaces are WHTs:*
$\mathscr{H}$ = class of all Hausdorff spaces
$\mathscr{M}$ = class of all metrizable spaces
$\mathscr{K}$ = class of all compact spaces
$\mathscr{N}$ =class of all normal spaces

Definition 2.1.3 (AE and ANE). *A closed subspace A in a topological space X has the* extension property *in X with respect to a space Y if and only if every map $f : A \to Y$ can be extended over X.*

A closed subspace A of a topological space X has the neighborhood extension property *in X with respect to Y if and only if every map $f : A \to Y$ can be extended over some open subspace $U \subset X$. (U may depend on f).*

Let $\mathscr{C}$ be a WHT.

An absolute extensor *(AE) for $\mathscr{C}$ is a space Y such that every closed subspace A of any space X in $\mathscr{C}$ has the extension property in X with respect to Y.*

An absolute neighborhood extensor *(ANE) for $\mathscr{C}$ is a space Y such that every closed subspace A of any space X in $\mathscr{C}$ has the neighborhood extension property in X with respect to Y.*

Definition 2.1.4 (AR and ANR). *Let $\mathscr{C}$ be a WHT.*

A retract of a topological space X is a space $A \subset X$ such that the identity map $Id : A \to A$ has a continuous extension $f : X \to A$.

A neighborhood retract of a topological space X is a space $A \subset X$ such that A is a retract of an open subspace $U \subset X$.

An absolute retract (AR) *for the class $\mathscr{C}$ is a space $Y \in \mathscr{C}$ such that every homeomorphic image of Y as a closed subspace of a space $Z \in \mathscr{C}$ is a retract of Z.*

An absolute neighborhood retract (ANR) *for the class $\mathscr{C}$ is a space $Y \in \mathscr{C}$ such that every homeomorphic image of Y as a closed subspace of a space $Z \in \mathscr{C}$ is a neighborhood retract of Z.*

The following proposition trivially holds:

Proposition 2.1.5. *Every AR for a WHT $\mathscr{C}$ is an ANR for $\mathscr{C}$.*
Let $\mathscr{D}$ be a WHT contained in $\mathscr{C}$ and let Y be a space in $\mathscr{D}$. If Y is an ANR/AR for $\mathscr{C}$ then Y is an ANR/AR for $\mathscr{D}$.
If $Y = \{p\}$ is a singleton then Y is an AR for every class $\mathscr{C}$ which contains a singleton space (and hence also contains Y). $\square$

Another well-known result is Tietze's extension theorem:

Theorem 2.1.6 (Tietze's extension theorem). *The interval $I = [0, 1]$ is an AE for the class $\mathscr{N}$ of all normal spaces.* $\square$

The following is also a useful result:

Proposition 2.1.7. *Any topological product of AEs for a class $\mathscr{C}$ is also an AE for $\mathscr{C}$.*

Proof. Let $\{Y_i : i \in I\}$ denote a family of AE's for the class $\mathscr{C}$, and let Y denote the topological product of the Y_i. Assume that X is an element of the class $\mathscr{C}$, that A is a closed subspace of X and that $f : A \to Y$ is a mapping. For all $i \in I$ define the canonical projection $p_i : Y \to Y_i$ and consider the composition

$$p_i \circ f : A \to Y_i.$$

Since Y_i is an AE for $\mathscr{C}$ there is an extension $g_i : X \to Y_i$. We may define a mapping $g : X \to Y$ by setting

$$p_i(g(x)) = g_i(x) \ \forall x \in X$$

It follows that $g|_A = f$ and so Y is an AE for $\mathscr{C}$. $\square$

We may generalize Tietze's extension theorem using the proposition above:

Corollary 2.1.8. *Any topological power of the unit interval I, such as I^n or the Hilbert cube, is an AE for the class $\mathscr{N}$ of normal spaces.* $\square$

The following will also prove useful:

Corollary 2.1.9. *The n-cell $\overline{B}^n$, the standard n-simplex Δ_n and any closed n-simplex σ of any polytope is an AE for $\mathscr{N}$.*

Proof. All of the spaces mentioned above are homeomorphic to I^n. $\square$

2.2 Polytopes

References: [5], [8]

Polytopes are a certain kind of spaces which have nice topological properties and which will be used extensively when dealing with coverings for instance when proving results about retracts. This section contains the basic definitions and properties of polytopes.

Definition 2.2.1 (Simplicial complex). *An* abstract simplicial complex *K is a pair $(\mathscr{V}, \Sigma)$, where $\mathscr{V}$ is a set of elements called* vertices *and Σ is a collection of finite subsets of $\mathscr{V}$ called* simplexes *with the property that each element of $\mathscr{V}$ lies in some element of Σ and, if $\sigma \in \Sigma$ then for every subset $\sigma' \subset \sigma$ it is true that $\sigma' \in \Sigma$. A simplex containing exactly the vertices $a_0, a_1, ...a_n$ is sometimes denoted $\{a_0, a_1, ..., a_n\}$.*

An abstract simplicial complex is infinite *if the set $\mathscr{V}$ is infinite. If it is not infinite, it is* finite. *The dimension of a simplex σ is defined by*

$$\dim(\sigma) = \text{ (number of vertices in } \sigma) - 1,$$

and the dimension of an abstract simplicial complex K is

$$\dim(K) = \sup\{\dim(\sigma) : \sigma \in \Sigma\}.$$

If L is a simplicial complex such that each vertex of L is also a vertex of K, and each simplex of L is also a simplex of K, then L is a subcomplex *of K.*

For most purposes we will in fact denote by K also the sets of vertices of K and the set of simplexes of K, so that in the definition of the simplicial polytope $|K|$ associated to K below, the domain of the map $\alpha : K \to I$ actually the set $\mathscr{V}$ of vertices of K. Similarly, a vertex v of K is often denoted as a vertex $v \in K$ and a simplex σ of K is often denoted as a simplex $\sigma \in K$.

Example 2.2.2. *If σ is a simplex, then the set $\dot{\sigma}$ of all proper subsimplices of σ is a simplicial complex.*

Remark 2.2.3. *Note that a simplicial complex of dimension ∞ is infinite, while an infinite complex may have finite dimension. For example, the simplicial complex $(\mathbb{Z}, \{\{n\} : n \in \mathbb{Z}\})$, where the only simplices are the vertices themselves, is an infinite complex of dimension 0.*

Example 2.2.4 (The nerve of a covering). *Let X be a topological space and let $\mathfrak{U} = \{U_\alpha \neq \emptyset : \alpha \in \mathscr{A}\}$ be a covering of X. Now let each $\alpha \in \mathscr{A}$ be a vertex in a simplicial complex denoted $\mathscr{N}$ which is constructed in the following way:*
$\{\alpha_0, \alpha_1, ...\alpha_n\}$ is a simplex of $\mathscr{N}$ if and only if $U_{\alpha_0} \cap U_{\alpha_1} \cap ... \cap U_{\alpha_n} \neq \emptyset$.

It is clear from the definitions that $\mathscr{N}$ is a simplicial complex, and it is called the nerve *of the covering $\mathfrak{U}$.*

If we let K be any nonempty simplicial complex, we may define a new set $|K|$ which is the set of all functions

$$\alpha : K \to I$$

such that

(a) For any $\alpha \in |K|$, $\{v \in K : \alpha(v) \neq 0\}$ is a simplex of K - in particular, $\alpha(v) \neq 0$ for only finitely many $v \in K$.

(b) For any $\alpha \in |K|, \sum_{v \in K} \alpha(v) = 1$.

The set $|K|$ is called the *simplicial polytope* associated with the simplicial complex K, and if L is a subcomplex of K, then $|L|$ is a *subpolytope* of $|K|$.

The polytope associated with the nerve of a covering is called the *geometric nerve* of the covering.

In order to define a topology on a given polytope we need the notion of a geometric simplex.

Definition 2.2.5 (Geometric simplex, the standard n-simplex in $\mathbb{R}^{n+1}$.). *Let $A = \{a_0, a_1, ...a_k\}$ be a set of geometrically independent points in $\mathbb{R}^n$, i.e. no $(k-1)$-dimensional hyperplane contains all the points. The* geometric k-simplex in $\mathbb{R}^n$ *(denoted σ^k) spanned by A is the convex hull*

$$\{\sum_{i=0}^{k} \lambda_i a_i \text{ where each } \lambda_i \in \mathbb{R}_+ \text{ and } \sum_{i=0}^{k} \lambda_i = 1\}.$$

of the set A, and the points of A are the vertices of σ^k. The simplex is also denoted $\sigma^k = (a_0, a_1, ...a_k)$. The set of all points $x \in \sigma^k$ for which each $\lambda_i > 0$ is the open geometric k-simplex spanned by A. *A simplex σ^m is a* face *or a* subsimplex *of the simplex σ^k if all the vertices of σ^m are also vertices of σ^k.*

The standard n-simplex in $\mathbb{R}^{n+1}$, *denoted Δ_n, is the geometric simplex spanned by the standard vectors $e_i = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}^{n+1}$ with the 1 in the i^{th} place, $i = 0, 1, ..., n$.*

If σ is a simplex in a simplicial complex K, then the corresponding *closed simplex* $|\sigma|$ is a subset of $|K|$ defined by

$$|\sigma| = \{\alpha \in |K| : \alpha(v) \neq 0 \Rightarrow v \in \sigma\}.$$

Proposition 2.2.6. *For every q-simplex σ in a simplicial complex K, the corresponding closed simplex $|\sigma|$ is in $1-1$ correspondence with the standard q-simplex Δ_q in $\mathbb{R}^{q+1}$.*

Proof. Let $v_0, ..., v_q$ be the vertices of σ and let $r_0, ..., r_q$ denote the vertices $(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 0, 1)$ of Δ_q. Define a function $f : \Delta_q \to |\sigma|$ by

$$f : \sum_{i=0}^{q} t_i r_i \mapsto \alpha \qquad \text{where} \qquad \alpha(v_i) = t_i \quad \forall \quad i = 0, 1, ..., q.$$

The points $t_i = \alpha(v_i)$ are called the *barycentric coordinates* of the point α in $|\sigma|$. Next define a function $g : |\sigma| \to \Delta_q$ by

$$g : \alpha \mapsto \sum_{i=0}^{q} \alpha(v_i) r_i \quad \forall \quad \alpha \in |\sigma|.$$

It is clear that $f \circ g = id_{|\sigma|}$ and $g \circ f = id_{\Delta_q}$ and hence f is a bijection. □

Now that we have a bijection f between each closed simplex $|\sigma|$ and the standard n-simplex Δ_n in $\mathbb{R}^{n+1}$ for some $n \in \mathbb{N}_0$, we may define a topology on $|\sigma|$. Assume that Δ_n has the Euclidean topology for all $n \in \mathbb{N}$ (induced from the usual topology on $\mathbb{R}^{n+1}$ as a subset). Then let a subset U be open in the closed q-simplex $|\sigma|$ if and only if $f(U)$ is open in Δ_q - that is, we let $|\sigma|$ have the only topology which makes f a homeomorphism.

We then say that $|\sigma|$ has the *Euclidean topology.*

Next we wish to define a topology on the polytope $|K|$ associated with a simplicial complex K, and we will require the topology to satisfy two conditions:

(PT1) Every subpolytope of $|K|$ is a closed subset of $|K|$.

(PT2) Every finite subpolytope $|L|$ of $|K|$, considered as a subspace of $|K|$, has the *Euclidean topology*, or in other words, its topology equals the subset topology when $|L|$ is considered to be a subset of the closed simplex $|\sigma|$, where σ is a simplex whose vertices are all the vertices of L (That is, σ is not necessarily a simplex of K.)

One topology which fulfills these requirements is the *Whitehead topology* $\mathscr{T}_w$ (usually referred to as the *weak topology*), which is defined as follows:

A set $U \subset |K|$ is open (or closed) if and only if, for every closed simplex $|\sigma|$ of $|K|$, the intersection $U \cap |\sigma|$ is an open (or closed) subset of $|\sigma|$. This is then the topology *coinduced* by the inclusion maps $i_\sigma : |\sigma| \to |K|$ for each simplex σ of K.

Always when talking about simplicial polytopes, it will be understood that it has the Whitehead topology unless otherwise is stated.

Proposition 2.2.7. *A subpolytope $|L|$ of a simplicial polytope $|K|$ is a closed subset of $|K|$. In particular, a closed simplex $|\sigma|$ is a closed subset of $|K|$.*

Proof. Let σ be a simplex in K; now, for each simplex σ' in L the intersection $\sigma' \cap \sigma$ is either empty or a subsimplex of σ. Since σ contains only finitely many subsimplices, the set

$$\{\sigma' \cap \sigma : \sigma' \text{ is a simplex in } L, \sigma' \cap \sigma \neq \emptyset\} \;=\; \{\sigma_i : i \in I\}$$

is a finite set of simplices.

Hence $|\sigma| \cap |L| = \bigcup_{i \in I} |\sigma_i|$ where I is a finite index set.

Now, if Δ_n is the standard n-simplex homeomorphic to $|\sigma|$ then each $|\sigma_i|$ is homeomorphic to some subsimplex of Δ_n, which is a closed subset, and hence $|\sigma_i|$ is a closed subset of $|\sigma|$. Hence, as a finite union of closed subsets,

$|\sigma| \cap |L|$ is a closed subset of $|\sigma|$.

Hence $|L|$ is closed in $|K|$.

$\square$

It follows that $\mathscr{T}_w$ fulfills the condition (PT1).

Proposition 2.2.8. *Let $|K|$ be any simplicial polytope, and let $|L|$ be a finite subpolytope of $|K|$. Then $|L|$ has the Euclidean topology.*

Proof. Denote by $v_0, v_1, ..., v_n$ the vertices of L and denote by σ the simplex spanned by the v_i (not necessary a simplex of K). Now σ is homeomorphic to the standard n-simplex Δ_n in $\mathbb{R}^{n+1}$. The topology on each simplex $\sigma \in |L|$ is then the relative topology from $\mathbb{R}^{n+1}$.

Now if $U \subset |L|$ is open in the "relative" topology on $|L|$ from $\mathbb{R}^{n+1}$, then it is clear that U is open in $|L|$ with the Whitehead topology. Conversely, if $U \subset |L|$ is open in $|L|$ with the Whitehead topology then $U \cap |\sigma|$ is open in $|\sigma|$ for each closed simplex $|\sigma| \in |L|$. Then $|L| \setminus U = \bigcup_{i=1}^{n} |\sigma_i| \setminus U$ which is closed in the relative topology since $|\sigma_i| \setminus U$ is closed in the relative topology for each $i \in \{1, ..., n\}$ where the σ_i are the simplices of L. Hence U is open in $|L|$ with the relative topology, and hence the relative topology from σ, or in other words, the Euclidean topology on $|L|$, and the Whitehead topology on $|L|$ are the same. $\square$

Now we have shown that $\mathscr{T}_w$ fulfills (PT2) as well.

Proposition 2.2.9. *Let the simplicial polytope $|K|$ have the Whitehead topology, and let X be a topological space. A function*

$$f : |K| \to X$$

is continuous if and only if $f|_{|\sigma|} : |\sigma| \to X$ is continuous for for every $\sigma \in K$.

Proof. "$\Rightarrow$" Trivial, since the restriction of a continuous map is always continuous.

"$\Leftarrow$" Let U be an open subset of X. Now

$$f^{-1}(U) \cap |\sigma| = (f|_{|\sigma|})^{-1}(U)$$

is open in $|\sigma|$ for every $\sigma \in K$, and hence f is continuous.

$\square$

Definition 2.2.10 (The metric topology). *Another topology which also satisfies (PT 1-2) is the* metric topology $\mathscr{T}_d$. *We may define a metric d on $|K|$ by setting*

$$d(\alpha, \beta) = \sqrt{\sum_{v \in \mathscr{V}} [\alpha(v) - \beta(v)]^2}.$$

The polytope associated to K with the metric topology will from now on be denoted $|K|_d$, while $|K|$ is used for the polytope with the Whitehead topology.

Clearly, the topology of a closed simplex as a subset of a simplicial polytope induced by the metric topology is the Euclidean topology. Hence, if a subset A of a simplicial polytope $|K|$ is open in the metric topology, then $A \cap |\sigma|$ is open in $|\sigma|$ for every closed simplex $|\sigma|$ in $|K|$ and hence A is open also in the Whitehead topology. It follows that $\mathscr{T}_d \subset \mathscr{T}_w$.

The following proposition is then obvious:

Proposition 2.2.11. *The identity map*

$$id : |K| \to |K|_d$$

is continuous.

Corollary 2.2.12. *A simplicial polytope $|K|$ with the Whitehead topology is a Hausdorff space.*

Proof. The metric space $|K|_d$ is Hausdorff, and the identity map $Id : |K| \to |K|_d$ is continuous - hence since two points $a \neq b$ have two disjoint open neighborhoods in the metric topology, the same two disjoint sets are also neighborhoods in the Whitehead topology. □

We have shown that $\mathscr{T}_d \subset \mathscr{T}_w$, but the opposite is not generally true - consider for instance the simplicial complex $K = (\mathbb{N}_0, \Sigma)$ where $\Sigma = \{\{n\} : n \in \mathbb{N}_0\} \cup \{\{0, n\} : n \in \mathbb{N}\}$. Now if σ^n is the closed simplex of the polytope $|K|$ corresponding to the abstract simplex $\{0, n\}$ then σ^n is homeomorphic to $[0, 1]$ by the homeomorphism that takes each point to its barycentric coordinate with respect to the vertex n. We may call this homeomorphism h_n. Now, if we denote

$$A = \bigcup_{n \in \mathbb{N}} h^{-1} \left([0, \frac{1}{n}) \right)$$

then A is open in the Whitehead topology since $A \cap \sigma^n = h_n^{-1} \left([0, \frac{1}{n}) \right)$ is open in σ^n and $A \cap \{n\}$ is either empty or $\{n\}$ (and hence open in $\{n\}$) for each 0-simplex $\{n\}$ of $|K|$.

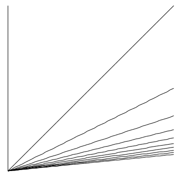

Figure 2.1: The set underlying the polytope $|K|$ may be visualized like this.

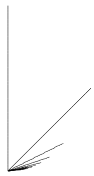

Figure 2.2: The subset A of $|K|$ then corresponds to a set which one may visualize like this.

However, A will not be open in the metric topology $\mathscr{T}_d$ on $|K|$ since for each $r > 0$

$$B_d(0, r) = \bigcup_{n \in \mathbb{N}} h_n^{-1}([0, r))$$

will contain points from $|K| \setminus A$. It follows that $\mathscr{T}_w \nsubseteq \mathscr{T}_d$.

Proposition 2.2.13. *For a simplicial complex K, the polytope $|K|$ is normal.*

Proof. Claim: $|K|$ is normal $\Leftrightarrow$ if A is a closed subset of $|K|$ then any map $f : A \to I$ can be continuously extended over $|K|$.

"$\Rightarrow$"
By Tietze's extension theorem.

"$\Leftarrow$"
Let A and B be two disjoint closed subsets of $|K|$. Define a function $f : A \cup B \to I$ by setting

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

Now f is continuous, and so by the assumption it has a continuous extension $g : |K| \to I$. Define

$$V = g^{-1}([0, \frac{1}{2}[),$$

$$U = g^{-1}(]\frac{1}{2}, 1]);$$

then V and U are disjoint neighborhoods of A and B, respectively. Hence $|K|$ is normal, and the claim holds. $\square$

To show that $|K|$ really is normal we then show the right hand side of the equivalence above. Let A be any closed subset of the simplicial polytope $|K|$, and let $f : A \to I$ be any continuous map. By Proposition (2.2.9) a continuous extension over $|K|$ exists if and only if there exists a family of maps $\{f_\sigma : |\sigma| \to I : \sigma \text{ is a simplex in } K\}$ such that

(a) if σ' is a face of σ, then $f_\sigma|\sigma' = f_{\sigma'}$

(b) $f_\sigma|(A \cap |\sigma|) = f|(A \cap |\sigma|)$.

We will use induction on the dimension of σ to prove that such a family exists.

If $\dim(\sigma) = 0$ then $|\sigma|$ is a singleton set, and so

- if $|\sigma| \subset A$ then define $f_\sigma = f|\ |\sigma|$
- if $|\sigma| \not\subseteq A$ then f_σ may take any value.

Let $q > 0$ and assume that f_σ is defined for all simplexes σ of dimension less than q, such that (a) and (b) hold. Let σ be a q-simplex, and define a function $f'_\sigma : |\dot{\sigma}| \cup (A \cap |\sigma|) \to I$ by setting

$$f'_\sigma|_{|\sigma'|} = f_{\sigma'} \qquad \text{if } \sigma' \text{ is a proper face of } \sigma$$

$$f'_\sigma|(A \cap |\sigma|) = f|(A \cap |\sigma|)$$

where $\dot{\sigma}$ is the simplicial complex consisting of all proper faces of σ. Now $\{f_{\sigma'} : \dim \sigma' < q\}$ is a family of maps satisfying both conditions (a) and (b), and hence f'_σ is a continuous map

$$|\dot{\sigma}| \cup (A \cap |\sigma|) \to I,$$

where $|\dot{\sigma}| \cup (A \cap |\sigma|)$ is a closed subset of $|\sigma|$. Since $|\sigma|$ is homeomorphic to some standard n-simplex Δ_n which is, as a closed subset of the normal space

$\mathbb{R}^n$, normal, it follows that $|\sigma|$ is also normal and so by Tietze's extension theorem, there exists a continuous extension

$$f_\sigma : |\sigma| \to I$$

of f'_σ.

Thus f_σ satisfies the conditions (a)-(b) and so the theorem is proved.

$\square$

Definition 2.2.14 (Open simplex). *Given a simplex σ in a simplicial complex K, the* open simplex $\langle\sigma\rangle$ *in $|K|$ associated with σ is the set*

$$\langle\sigma\rangle = \{\alpha \in |K| : \alpha(v) \neq 0 \Leftrightarrow v \in \sigma\}.$$

As noted in Proposition (2.2.6) and the following discussion, for each closed simplex $|\sigma|$ in $|K|$ there is a homeomorphism

$$f : \Delta_n \to |\sigma| \qquad \text{for some} \qquad n \in \mathbb{N}.$$

Then, clearly, $\langle\sigma\rangle = f(Int(\Delta_n))$.

An open simplex does not have to be open in $|K|$ - for instance, if K has three vertices and it contains all possible simplexes then $|K|$ is homeomorphic to Δ_2 and if σ is a simplex containing two vertices then $\langle\sigma\rangle$ is homeomorphic to one of the sides of Δ_2 minus the vertices - which is clearly not open in Δ_2 and hence $\langle\sigma\rangle$ is clearly not open in $|K|$.

However, since $\langle\sigma\rangle = |\sigma| \setminus |\dot{\sigma}|$, the open simplex $\langle\sigma\rangle$ is open in $|\sigma|$.

Each point $\alpha \in |K|$ belongs to a unique open simplex - $\langle s\rangle$, where $s = \{v \in K : \alpha(v) \neq 0\}$. Thus the open simplexes form a partition of $|K|$.

Proposition 2.2.15. *Let $A \subset |K|$. Then A contains a discrete subset which consists of exactly one point from each open simplex which meets A.*

Proof. For each simplex $\sigma \in K$ such that $A \cap \langle\sigma\rangle \neq \emptyset$ let $\alpha_\sigma \in A \cap \langle\sigma\rangle$ and let

$$A' = \{\alpha_\sigma : A \cap \langle\sigma\rangle \neq \emptyset\}$$

Since any simplex contains only a finite amount of subsimplexes, a closed simplex can only contain a finite subset of A' - thus every subset of A' is closed in the Whitehead topology and so A' is discrete.

$\square$

Corollary 2.2.16. *Every compact subset of $|K|$ is contained in the union of a finite number of simplexes.*

Proof. Let C be a compact subset of $|K|$ which is not contained in any finite union of simplexes. Then C meets infinitely many open simplexes. Then by Proposition (2.2.15) it contains an infinite discrete subset A' which, since it is closed, is compact also. Let $\mathscr{A} = \{V_a : a \in A'\}$ be a set of open sets such that $V_a \cap A' = \{a\} \quad \forall \quad a \in A'$. Now $\mathscr{A}$ is an open covering of A' which has no finite subcovering - which gives a contradiction. $\square$

Corollary 2.2.17. *A simplicial complex K is finite if and only if the set $|K|$ is compact.*

Proof. "$\Rightarrow$"
Each closed simplex is homeomorphic to some standard n-simplex Δ_n which is compact, hence every simplex is compact. The set $|K|$ is then compact since it is the finite union of compact sets.

"$\Leftarrow$"
Cor (2.2.16)

$\square$

Definition 2.2.18 (The open star of a vertex). *The open star $St(v)$ of a vertex v in a simplicial polytope $|K|$ is defined as*

$$St(v) = \{\alpha \in |K| : \alpha(v) \neq 0\}$$

The mapping

$$g : |K|_d \to I \qquad \text{given by} \qquad \alpha \mapsto \alpha(v)$$

is continuous, and hence $St(v)$ is an open subset of $|K|_d$ and hence also of $|K|$. It follows that

$$\alpha \in St(v) \Leftrightarrow \alpha(v) \neq 0 \Leftrightarrow \alpha \in \langle\sigma\rangle \quad \text{where} \quad v \in \sigma$$

and thus

$$St(v) = \bigcup\{\langle\sigma\rangle : v \text{ is a vertex of } \sigma\}$$

Conversely, the *closed star* $\overline{St}(v)$ of a vertex v is the union of all closed simplexes which have v as a vertex.

Remark 2.2.19. *From here on, the polytope associated with a simplicial complex K will be denoted K also, when there is no danger of confusion.*

2.3 Dugundji's extension theorem

In this section we will prove a theorem by Dugundji on the extension property of mappings $f : A \to L$ to the space $X \supset A$ where X is metrizable, A is a closed subset of X and L is a locally convex topological linear space. When dealing with metrizable spaces, this is more general and hence more useful than the well-known Tietze's extension theorem.

Definition 2.3.1 (Canonical covering). *Let X be a topological space, and let A be a closed subspace of X. A covering of $X \backslash A$ by a collection γ of open sets of $X \backslash A$ is called a* canonical covering *of $X \backslash A$ if and only if the following conditions hold:*

(CC1) γ is neighborhood-finite *(see Definition (1.4.3))*

(CC2) Every neighborhood of any boundary point of A in X contains infinitely many elements of γ.

(CC3) For each neighborhood V of a point $a \in A$ in X there exists a neighborhood W of a in X, $W \subset V$ such that every open set $U \in \gamma$ which meets W is contained in V.

Example 2.3.2. *Let $X = B(0,1)$ have the Euclidean topology and let $A = \{0\}$. Denote by U_n the set $\{z \in X : \frac{1}{n+2} < d(0,z) < \frac{1}{n}\}$. Now $\gamma = \{U_n : n \in \mathbb{N}\}$ is a covering of $X \setminus A$ and since each U_n only intersects two others, γ is neighborhood-finite. The conditions (CC2) and (CC3) are trivially fulfilled. Hence γ is a canonical covering of $X \setminus A$.*

In the example above, X was a metric space. We will now see that for any metric space such a covering can be found.

Lemma 2.3.3. *If X is a metrizable space and A is a (proper) closed subspace of X, then there exists a canonical covering of $X \setminus A$.*

Proof. Let d be a metric defining the topology in X. For each $x \in X \backslash A$ let S_x denote the open neighborhood of x in X defined by

$$S_x = B_X(x, \frac{1}{2}d(x, A))$$

Hence $\{S_x : x \in X \backslash A\}$ is an open covering of $X \backslash A$. By Thm (1.5.13), since $X \backslash A$ is metrizable it is paracompact. Thus the open covering $\{S_x : x \in X \backslash A\}$ has a locally finite open refinement γ.

We now wish to show that γ satisfies (CC2) and (CC3).

Let V be any neighborhood of an arbitrary point $a \in A$ in X. Then there exists $k \in \mathbb{R}_+$ such that $B_X(a, 2k) \subset V$.

Denote by W the neighborhood of a defined by

$$W = B_X(a, \frac{1}{2}k)$$

Now assume that $U \in \gamma$ meets W at some point $y \in X$. Since γ is a refinement of $\{S_x : x \in X \backslash A\}$ there must be a point $x \in X \backslash A$ such that $y \in U \subset S_x$. Hence by the definition of S_x it follows that

$$d(a,x) \leq d(a,y)+d(y,x) < \frac{1}{2}k+\frac{1}{2}d(x,A) \leq \frac{1}{2}k+\frac{1}{2}d(a,x) \Rightarrow \frac{1}{2}d(a,x) < \frac{1}{2}k.$$

Hence $d(a,x) < k$.

Since for any $z \in S_x$

$$d(x,z) < \frac{1}{2}d(x,A) \leq \frac{1}{2}d(x,a)$$

we have

$$d(a,z) \leq d(a,x) + d(x,z) < d(a,x) + \frac{1}{2}d(a,x) = \frac{3}{2}d(a,x) < 2k.$$

It follows that $z \in V$, hence $U \subset S_x \subset V$ and so (CC3) holds.

Now assume that $a \in \partial A$. To prove that a neighborhood V of a contains infinitely many open sets of γ, it is enough to show that V contains a set $U_0 \in \gamma$ and a neighborhood V_0 of a which does not meet U_0. (Then this V_0 contains a set $U_1 \in \gamma$ and a neighborhood V_1 of a such that $U_1 \cap V_1 = \emptyset$, and by continuing this procedure we obtain a sequence $\{U_0, U_1, U_2, ...\}$ of sets $U_i \in \gamma$ where $U_i \subset V \ \forall\ i \in \mathbb{N}$.)

Let V be any neighborhood of a. Because $a \in \partial A$, V contains a point $y \in X \setminus A$. Hence there exists $U \in \gamma$ such that $y \in U$. Let k be such that $B_X(a, 2k) \subset V$. Now, by the same argument as above, there exists a point $x \in X \setminus A$ such that $d(a,x) = k' < k$ and $U \subset S_x \subset V$.

Now let V_0 denote the neighborhood of a defined by

$$V_0 = B_X(a, \frac{1}{2}k')$$

Then $V_0 \subset V$ and $V_0 \cap U = \emptyset$, since

$$\begin{aligned} u \in U \subset S_x \quad &\Rightarrow d(x,u) < \tfrac{1}{2}d(x,A) \leq \tfrac{1}{2}d(x,a) = \tfrac{1}{2}k' \\ &\Rightarrow d(u,a) \geq d(x,a) - d(x,u) > k' - \tfrac{1}{2}k' = \tfrac{1}{2}k' \\ &\Rightarrow u \notin V_0 \end{aligned}$$

and so (CC2) holds.

$\square$

Lemma 2.3.4 (Replacement by polytopes). *If X is a metrizable space and A is a closed proper subspace of X then there exists a space Y and a map*

$$\mu : X \to Y$$

with the following properties:

(RP1) The restriction $\mu|A$ is a homeomorphism of A onto a closed subspace $\mu(A)$ of Y.

(RP2) The open subspace $Y \setminus \mu(A)$ of Y is an infinite simplicial polytope with the Whitehead topology, and

$$\mu(X \setminus A) \subset Y \setminus \mu(A)$$

(RP3) Every neighborhood of a boundary point of $\mu(A)$ in Y contains infinitely many simplexes of the simplicial polytope $Y \setminus \mu(A)$.

Proof. Let γ be a canonical covering of $X \setminus A$ and let N be the geometric nerve of γ (We may assume that $\emptyset \notin \gamma$). Then the vertices of N are in 1-1 correspondence with the open sets in γ. Denote by v_U the vertex of N corresponding to $U \in \gamma$.

Let Y denote the disjoint union $A \dot{\cup} N$, and topologize Y as follows:

Let $y \in Y$ be an arbitrary point. If $y \in N$, take as a basis for neighborhoods of y in Y all of the neighborhoods of y in N. If $y \in A$, take as a basis for neighborhoods or y in Y all of the sets V^* defined by: If V is an arbitrary neighborhood of y in X, then V^* is a set in Y consisting of the points of $V \cap A$ and the points of the open stars $St(v_U)$ in N, where U is an element of γ contained in V.

Claim: The bases for neighborhoods described above define a topology on Y.

Proof: Denote

$$\mathscr{B} = \{U, V^* : U \text{ is a neighborhood of } y \in N \text{ in } N, V \text{ is a neighborhood of } y' \in A \text{ in } X\}$$

We will show that $\mathscr{B}$ defines a basis for some topology on Y, and that in this topology the original bases for neighborhoods really are bases for neighborhoods.

Clearly $\mathscr{B}$ covers Y. It now suffices to show that given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathscr{B}$ such that $x \in B \subset B_1 \cap B_2$.

If B_1 and B_2 are both open neighborhoods in N of some point $y \in N$, then $B_1 \cap B_2$ is also an open neighborhood of y in N and we may set $B = B_1 \cap B_2$.

If B_1 and B_2 can both be written

$$B_1 = V_1^*, \qquad B_2 = V_2^*$$

where V_1 and V_2 are open subsets of X intersecting A, then

$$x \in V_1^* \cap V_2^*$$

means:

i) If $x \in A$: $x \in (V_1 \cap A) \cap (V_2 \cap A) = (V_1 \cap V_2) \cap A$

ii) If $x \in N$: $x \in St(v_{U_1}) \cap St(v_{U_2})$ where $U_1, U_2 \in \gamma, U_1 \subset V_1$, and $U_2 \subset V_2$.

In the case *i)*, $V_1 \cap V_2$ is a neighborhood of x in X. If $y \in (V_1 \cap V_2)^* \cap N$ then $y \in St(v_U)$ where $U \subset (V_1 \cap V_2)$; hence $U \subset V_1$ and $U \subset V_2$, thus $St(v_U) \subset V_1^*$, and $St(v_U) \subset V_2^*$, hence $y \in St(v_U) \subset V_1^* \cap V_2^*$. In other words, we may set $B = (V_1 \cap V_2)^*$.

In the case *ii)*, since open stars of vertices of N are open sets of N, we may set $B = St(v_{U_1}) \cap St(v_{U_2}) \in \mathscr{B}$.

Finally, consider the case where $B_1 = U$ which is an open subset of N and $B_2 = V^*$ for some open set $V \subset X$, and suppose that $y \in U \cap V^*$.

Now, since $U \cap V^* \subset N$, we have $y \in U \cap St(v_{U'}) \subset U \cap V^*$ for some $U' \in \gamma$ which is open in N. Thus we may choose $B = U \cap St(v_{U'}) \in \mathscr{B}$.

We have now shown that $\mathscr{B}$ is a basis for some topology on Y, and it is easy to show that the initially defined bases of neighborhoods are bases of neighborhoods in this topology. It follows that the topology whose basis is $\mathscr{B}$ is the correct one. $\square$

Claim: Y with this topology is a Hausdorff space, and both A and N preserve their original topologies as subspaces of Y.

Proof: If a and y are two different points of Y so that they are both in N then, since a simplicial polytope is Hausdorff, they have disjoint neighborhoods V and U in N. Now these are also open in Y.

If a and y are both in A then, since X is metric, they have disjoint neighborhoods V_a and V_y in X. Since γ is a canonical covering, then by the condition (CC3) there exist neighborhoods W_a and W_y such that every $U \in \gamma$ which meets W_a is contained in V_a, and every $V \in \gamma$ which meets W_y is contained in V_y. We will show that the neighborhoods W_a^* and W_y^* are disjoint.

Clearly, $(W_a \cap A) \cap (W_y \cap A) = \emptyset$. If $U_a \in \gamma$ such that $U_a \subset W_a$, then each $V \in \gamma$ which meets U_a meets W_a and hence is contained in V_a. Similarly if we replace a with y. Hence if σ_a is a simplex in N with v_{U_a} as a vertex, and σ_y is a simplex in N with v_{U_y} as a vertex, then σ_a and σ_y have no vertices in common. Hence $\sigma_a \cap \sigma_y = \emptyset$ and so $St(v_{U_a}) \cap St(v_{U_y}) = \emptyset$. It follows that W_a^* and W_y^* are disjoint neighborhoods of a and y in Y.

Finally assume that $a \in A$ and $y \in N$. If $a \in Int_X(A)$ then $V = Int_X(A)$ is a neighborhood of a which does not meet any elements of γ. Hence $V^* = V$. It follows that if U is any neighborhood of y in N then V and U are disjoint neighborhoods of a and y in Y.

Thus let $a \in \partial_X A$. Let σ be the open simplex of N containing y. (by a previous comment, the open simplexes of N constitute a partition of N), and denote its vertices by $v_0, v_1, ..., v_n$. They then correspond to open sets $U_0, U_1, ..., U_n \in \gamma$, where $U_i \subset X \setminus A \quad \forall \quad i = 0, 1, ..., n$. Choose a neighborhood V of a in X such that $U_i \nsubseteq V \ \forall \ i = 0, 1, ..., n$. (Choose a point $x_i \in U_i \ \forall \ i = 0, 1, ..., n$ and let $V = B_X(a, r)$ where $r < d(a, x_i) \ \forall \ i = 0, 1, ..., n$)

From the condition (CC3) in the definition of a canonical covering there exists a neighborhood W of a such that each $U \in \gamma$ which meets W is contained in V. Then W cannot meet any of the sets U_i.

Now W^* is a neighborhood of a in Y. If $U \subset W$ for some $U \in \gamma$ then the open star $St(v_U)$ consists of all open simplexes of N which have v_U as a vertex. Let σ' be such a simplex of N. Then, if $v_{U'}$ is another vertex of σ', then

$$U \cap U' \neq \emptyset \Rightarrow W \cap U' \neq \emptyset \Rightarrow U' \neq U_i \quad \forall \quad i = 0, 1, ..., n.$$

Hence none of the v_i are vertices of σ', and so $\sigma \cap \sigma' = \emptyset$. In other words, since U was any element of γ contained in W and σ' was any simplex of N with v_U as a vertex, we get that $y \notin W^*$. By the same argument as above, if we let S denote the union of all closed stars $\overline{St}(v_U)$, then also $y \notin S$.

Any union S of closed stars of N is necessarily closed in N, since if s is a simplex of N then S intersects s in a collection of closed subsimplexes $|s'|$ where s' is a subsimplex of s, and any simplex s only has finitely many subsimplexes. Hence $S \cap |s|$ is in reality a finite union of closed subsets of $|s|$ and is hence closed in $|s|$. Because this is true for any simplex s of N, S is then a closed subset of N.

By the argument above, S is a closed subset of N. Let $W' = N \setminus S$. Now W' is an open neighborhood of y in N and thus in Y, and $W' \cap W^* = \emptyset$.

Conclusion: Y is Hausdorff! It is trivial that A and N preserve their original topologies as subspaces of Y. Hence the claim has been proved. $\square$

Since $N = \bigcup_{y \in N} V_y$, where each V_y belongs to the basis for neighborhoods of y, N is open in Y, and hence A is closed in Y.

Because γ is a neighborhood-finite covering of the metrizable space $X \setminus A$, we can define a canonical map

$$\kappa : X \setminus A \to N$$

as follows:

Let d be a metric in $X \setminus A$ which defines the topology of $X \setminus A$, and let $x \in X \setminus A$ be an arbitrary point. Since the covering γ is locally finite, x is contained in only a finite number of open sets of γ - denote these sets $U_0, U_1, ...U_n$. Let Δ denote the closed n-simplex in N corresponding to the vertices $v_{U_0}, ...v_{U_n}$. Then we define $\kappa(x)$ as the point in Δ with barycentric coordinates $\xi_0, \xi_1, ...\xi_n$ given by

$$\xi_i = \frac{d(x, X - U_i)}{\sum_{j=0}^{n} d(x, X - U_j)}$$

(see the proof of Proposition (2.2.6) for the definition of barycentric coordinates).

Now construct a function $\mu : X \to Y$ by setting:

$$\mu(x) = \begin{cases} x & \text{if } x \in A \\ \kappa(x) & \text{if } x \in X \setminus A \end{cases}$$

The function μ is continuous in X if it is continuous on the boundary of A in X. In order to prove the continuity on ∂A, let $a \in \partial A$ be arbitrary and let V^* be a basic neighborhood of $\mu(a)$ in Y. Then V^* is, by definition, determined by some neighborhood V of a in X, and by the condition (CC3) for canonical coverings such as γ, there exists a neighborhood $W \subset V$ of a in X such that every $U \in \gamma$ for which $U \cap W \neq \emptyset$ is contained in V. We will show that $\mu(W) \subset V^*$.

Let $x \in W$ be an arbitrary point - we are about to show that $\mu(x) \in V^*$. If $x \in A$, then

$$\mu(x) = x \in A \cap W \subset A \cap V \subset V^*$$

If $x \in X \setminus A$, then $\mu(x) = \kappa(x) \in N$. Since $\{St(v_U) : U \in \gamma\}$ covers N, there exists a set $U \in \gamma$ which is such that $\kappa(x) \in St(v_U)$. But then $\kappa(x) \in \langle\sigma\rangle$ where v_U is a vertex of σ, and from that it follows that $d(x, X \setminus U) > 0$, which gives $x \in U$. Hence U meets W at the point x, and thus $U \subset V$. It follows from the definition of V^* that $St(v_U) \subset V^*$.

Hence, for any basic neighborhood V^* of $\mu(a)$ in Y there exists a neighborhood W of a in X such that $\mu(W) \subset V^*$, or, in other words μ is continuous in ∂A and so it is continuous in X.

Now to the properties (RP1)-(RP3):

We have showed that N is open in Y, therefore A is closed and furthermore it keeps its original topology as a subspace of Y. Hence

$$\mu|A = Id_A : A \to A$$

is a homeomorphism. Thus (RP1) holds.

By definition $Y \setminus \mu(A) = Y \setminus A = N$ is a simplicial polytope with Whitehead topology. Furthermore, by the definition of a canonical covering γ, any neighborhood of any boundary point of A in X contains infinitely many elements of γ, hence γ must have infinitely many elements and so N is infinite. (RP2) holds.

Finally, each neighborhood V of a boundary point of A contains infinitely many elements U_i of γ (CC2) and hence V^* contains all the 0-dimensional simplexes $\{v_{U_i}\}$ which are infinitely many. Now (RP3) holds as well.

□

Definition 2.3.5 (Locally convex linear topological space). *A* linear topological space *is a real vector space L with a Hausdorff topology such that vector addition $x + y$ and scalar multiplication αx are continuous with respect to the Hausdorff topology on L and the usual topology on $\mathbb{R}$. L is* locally convex *if for each $a \in L$ and neighborhood U of a in L there exists a convex neighborhood V of a in L such that $a \in V \subset U$.*

Theorem 2.3.6 (Dugundji's extension theorem). *Let X be a metrizable space, A a closed subspace of X, L a locally convex topological linear space. If $f : A \to L$ is a mapping then there exists a continuous extension $g : X \to L$ of f such that $g(X)$ is contained in the convex hull of $f(A)$ in L.*

Proof. We will use the space $Y = A \cup N$ and the map $\mu : X \to Y$ which were constructed in Lemma (2.3.4). It will be enough to prove that the map $f : A \to L$ has an extension $F : Y \to L$ such that $F(Y)$ is contained in the convex hull of $f(A)$ in L, since the composition $g = F \circ \mu : X \to L$ will then be an extension of f for which $g(X)$ is contained in the convex hull of $f(A)$ in L.

Now let d be a metric which defines the topology in X, and let γ be the same canonical covering of $X \setminus A$ which was used to construct the space Y and the map μ in Lemma (2.3.4).

Let N^0 denote the set of all the vertices of N. We will define a map $\Phi : A \cup N^0 \to L$ in the following way:

In each open set $U \in \gamma$, select a point x_U and then pick a point $a_U \in A$ such that $d(x_U, a_U) < 2d(x_U, A)$. Define the map Φ by setting:

$$\begin{array}{ll} \Phi(a) = f(a) & \text{if } a \in A \\ \Phi(v_U) = f(a_U) & \text{if } v_U \in N^0 \end{array}$$

Claim: Φ is continuous

Proof: Because N^0 is an isolated set, $\Phi|N^0$ is trivially continuous. So, it suffices to check the continuity on ∂A ($\partial_X A = \partial_Y A = \partial_{A \cup N^0} A$) to show that Φ is continuous on $A \cup N^0$.

Let $a \in \partial_X A$ be an arbitrary point, and let M be any neighborhood of $\Phi(a) = f(a)$ in L. Because f is continuous, there exists a real number $\delta > 0$ such that $f(B_A(a, \delta)) \subset M$. Denote $V = B_X(a, \frac{\delta}{3})$, and let V^* be the basic neighborhood of $a = \mu(a)$ in Y as defined in Lemma (2.3.4). If we can show that

$$\Phi[V^* \cap (A \cup N^0)] \subset M$$

then the map Φ will be continuous in the point a, hence in ∂A and thus in all of $A \cup N^0$, since $V^* \cap (A \cup N^0)$ is a neighborhood of a in $A \cup N^0$.

Let y be any point in $V^* \cap (A \cup N^0)$. If $y \in A$, then

$$y \in V^* \cap A = V \cap A$$

and thus $d(a, y) < \frac{\delta}{3} < \delta$, which implies that $\Phi(y) = f(y) \in f(B_A(a, \delta)) \subset M$.

If $y \in N^0$, then $y = v_U$ for some $U \in \gamma$ where $U \subset V$. This implies that $d(a, x_U) < \frac{\delta}{3}$ and it follows that

$$d(a, a_U) \leq d(a, x_U) + d(x_U, a_U) < d(a, x_U) + 2d(x_U, A) \leq 3d(a, x_U) < \delta.$$

and thus $\Phi(v_U) = f(a_U) \in f(B_A(a, \delta)) \subset M$.

Hence Φ is continuous. $\square$

Extending Φ over Y: Since L is a linear space, we can extend linearly over each simplex of N the map Φ which is given on the vertices, obtaining a function $F : Y \to L$. Since addition and scalar multiplication in L are continuous, F is continuous on each simplex of N. It follows from Proposition (2.2.9) that F is continuous in all of N. Hence, again it suffices to check the continuity of F at the boundary points of A in Y.

Let $a \in \partial A$ be an arbitrary point and let M be any neighborhood of the point $F(a) = f(a)$ in L. Because L is locally convex, M contains a convex neighborhood K of $f(a)$ in L, and since Φ is continuous there exists a basic neighborhood V^* of a in Y such that $\Phi[V^* \cap (A \cup N^0)] \subset K$. Now, V^* is determined by a neighborhood V of a in X as in Lemma (2.3.4). By (CC3) there exists a neighborhood W of a in X such that $W \subset V$ and such that each $U \in \gamma$ which meets W is contained in V. This neighborhood W determines another basic neighborhood W^* of a in Y, and we will show that F is continuous by showing that $F(W^*) \subset K$.

Let $y \in W^*$. If $y \in A$, then

$$y \in W^* \cap A = W \cap A \subset V \cap A = V^* \cap A$$

and hence $F(y) = \Phi(y) \in K$.

If $y \in N$ then y is a point of some star $St(v_U)$ with $U \subset W$ by the definition of W^*. Since the open simplexes of N constitute a partition of N, the point y is an interior point of some simplex Δ of N, whose vertices can be taken to be $v_{U_0}, ..., v_{U_n}$. Because $y \in St(v_U)$, U is one of the open sets $U_0, ..., U_n$. By the definition of the nerve we must have that for each $U_i, i = 0, 1, ..., n$, U_i meets U and thus also W. Hence $U_i \subset V$ for all $i = 0, ..., n$ and thus all the vertices $v_{U_0}, ..., v_{U_n}$ are contained in $V^* \cap N^0$. Hence $\Phi(v_{U_i}) \in K$ for all $i = 0, ..., n$ and thus, since K is convex and F is linear on Δ (as a linear extension of Φ), it holds that $F(\Delta) \subset K$, and in particular, $F(y) \in K$. Because y was arbitrarily chosen, $F(W^*) \subset K$.

Now by definition, $\Phi(A \cup N^0) = f(A)$. Since F is obtained from Φ by linear extension, where the coefficients are always ≥ 0 and adding up to 1, it is clear that $F(Y)$ is contained in the convex hull of $\Phi(A \cup N^0) = f(A)$.

The proof is complete.

□

Corollary 2.3.7. *Every convex set in a locally convex topological linear space is an AE for the class $\mathscr{M}$ of metrizable spaces.*

Proof. Let Y be a convex set in a locally convex topological linear space L; let A be any closed subset of any metrizable space X and let $f : A \to Y$ be a mapping. Then by the previous theorem there exists a continuous extension $g : X \to L$ such that $g(X)$ is contained in the convex hull of $f(A)$ - which, since Y is convex is contained in Y. Thus we have a continuous extension $g : X \to Y$, and so Y is an AE for $\mathscr{M}$.

□

2.4 The Eilenberg-Wojdyslawski theorem

In this section we prove the Eilenberg-Wojdyslawski theorem, which enables us to use Dugundji's extension theorem when dealing with ANRs. Reference: [5]

Assume that Y is a metrizable space and that d is a bounded metric for Y.

Let $L = C(Y) = \{f : Y \to \mathbb{R} : f \text{ is bounded and continuous}\}$. Then, clearly, L forms a vector space over $\mathbb{R}$ with addition

$$(f+g)(y) = f(y) + g(y)$$

$$(\alpha f)(y) = \alpha(f(y))$$

for all $f, g \in L$, $\alpha \in \mathbb{R}$ and $y \in Y$.

We define a norm in L by setting

$$\|f\| = \sup_{y \in Y} |f(y)|.$$

Then, clearly, if $f, g \in L$ and $\alpha \in \mathbb{R}$,

$$\|f+g\| = \sup_{y\in Y}|f(y)+g(y)| \leq \sup_{y\in Y}(|f(y)|+|g(y)|) \leq \sup_{y\in Y}|f(y)|+\sup_{y\in Y}|g(y)| = \|f\|+\|g\|.$$

$$\|\alpha f\| = \sup_{y\in Y}|\alpha f(y)| = |\alpha| \sup_{y\in Y}|f(y)| = |\alpha|\ \|f\|.$$

$$\|f\| = 0 \Leftrightarrow \sup_{y\in Y}|f(y)| = 0 \Leftrightarrow |f(y)| = 0 \quad \forall\ y \Leftrightarrow f = 0.$$

Hence, $\|.\|$ is a norm.

Proposition 2.4.1. *L is Banach.*

Proof. Let (f_i) be a Cauchy sequence in L. Then the sequence $(f_i(x))$ is Cauchy in $\mathbb{R}$ for all $x \in Y$, and since $\mathbb{R}$ is complete the sequence converges. Hence we may define a function $f : Y \to \mathbb{R}$ by setting

$$f(x) = \lim_{i\to\infty} f_i(x)$$

Hence $f_i \to f$ pointwise. Furthermore, if we define

$$M_i = \sup_{y\in Y}|f_i(y)|$$

then the sequence (M_i) is also Cauchy in $\mathbb{R}$ and hence convergent - hence

$$M = \lim_{i \to \infty} M_i = \sup_{y \in Y} |f(y)|$$

and so f is bounded. Moreover, since $\|.\|$ is the sup-norm, it follows that $f_i \to f$ uniformly, and so f is continuous. Hence $f \in L$ and so L is complete. $\square$

Let $a \in Y$, and consider the bounded continuous function $f_a : Y \to \mathbb{R}$ defined by

$$f_a(y) = d(a, y) \ \forall \ y \in Y.$$

Clearly, $f_a \in L$. Denote

$$\chi : Y \to L; a \mapsto f_a$$

Lemma 2.4.2. *The function χ is an isometry, called the canonical isometric embedding of the bounded metric space Y into L.*

Proof. Let $a, b \in Y$. Then we have

$$d(a, b) = |f_a(b)| = |f_a(b) - f_b(b)| \leq \|f_a - f_b\| = \sup_{y \in Y} |d(a, y) - d(b, y)| \leq d(a, b).$$

Hence,

$$d(\chi(a), \chi(b)) = \|f_a - f_b\| = d(a, b).$$

Thus, χ is an isometry. $\square$

Theorem 2.4.3 (The Eilenberg-Wojdyslawski theorem). *Let Y be a bounded metric space. The image $\chi(Y)$ of the canonical isometric embedding $\chi : Y \to L$ into the Banach space $L = C(Y) = \{f : Y \to \mathbb{R} : f \text{ is bounded and continuous}\}$ is a closed subset of the convex hull Z of $\chi(Y)$ in L.*

Proof. It suffices to show that $Z \backslash \chi(Y)$ is open in Z. Let $g \in Z \backslash \chi(Y)$. Since Z is the convex hull of $\chi(Y)$ there exists a finite number of points $a_1, a_2, ..., a_n \in Y$ such that

$$g = \sum_{i=1}^{n} t_i f_i \text{ where } f_i = \chi(a_i), \ \sum_{i=1}^{n} t_i = 1, t_i \in \mathbb{R}_+ \ \forall \ i = 1, ..., n.$$

Since $g \notin \chi(Y)$ it follows that $g \neq f_i \quad \forall \quad i = 1, ..., n$. Choose $\delta \in \mathbb{R}$ such that $0 < \delta < \frac{1}{2}d(g, f_i) \quad \forall \quad i = 1, ..., n$ where d now denotes the metric in L defined by the norm.

Denote by V_δ the open neighborhood of g in Z given by

$$V_\delta = \{\Phi \in Z : d(g, \Phi) < \delta\}.$$

We now show that $V_\delta \subset Z \backslash \chi(Y)$, because then since g was arbitrarily chosen, $Z \backslash \chi(Y)$ must be open.

Assume that $y \in Y$ is such that $f = \chi(y) \in V_\delta$. By the choice of δ we have that

$$d(\chi(a_i), \chi(y)) = d(f_i, f) \geq d(g, f_i) - d(g, f) > 2\delta - \delta = \delta \qquad \forall \quad i = 1, ..., n.$$

Hence we obtain

$$d(g, f) = \|g - f\| \geq |g(y) - f(y)| = |g(y)| = \sum_{i=1}^{n} t_i f_i(y) > (\sum_{i=1}^{n} t_i)\delta = \delta.$$

It follows that $f \notin V_\delta$, which is a contradiction. Hence $V_\delta \subset Z \backslash \chi(Y)$ and so $Z \backslash \chi(Y)$ is open in Z. Hence $\chi(Y)$ is closed in Z.

□

2.5 ANE versus ANR

As mentioned before, it can be shown that for metrizable spaces, the concepts of AE/ANE and AR/ANR are essentially the same (in fact, Väisälä gives the definition of AE/ANE as AR/ANR). This section is devoted to showing exactly that.

Theorem 2.5.1. *Consider $\mathscr{M}$, the weakly hereditary class of metrizable spaces. Any space $Y \in \mathscr{M}$ is an ANE (or AE) for $\mathscr{M}$ if and only if it is an ANR (or AR) for $\mathscr{M}$.*

Proof. I will only include the proof for ANE/ANR. The proof for AE/AR is similar.

"$\Rightarrow$" Let Y be an ANE for $\mathscr{M}$, and let $h : Y \to Z_0$ be an arbitrary homeomorphism onto a closed subspace Z_0 of a space $Z \in \mathscr{M}$. Since Y is an ANE for $\mathscr{M}$ the map $h^{-1} : Z_0 \to Y$ has a continuous extension $g : U \to Y$ for some open neighborhood U of Z_0 in Z. Then $r = h \circ g : U \to Z_0$ is a retraction and hence Y is an ANR for $\mathscr{M}$.

"$\Leftarrow$" Let Y be an ANR for $\mathscr{M}$, and give Y a bounded metric (For every metric space X with metric d there is a bounded metric d' which is equivalent to d, i.e. (X, d) is homeomorphic to (X, d')) and consider the canonical isometric embedding $\chi : Y \to L = C(Y)$, where L is Banach. By Thm (2.4.3) the homeomorphic image $Z_0 = \chi(Y)$ is a closed subset of the convex hull Z of $\chi(Y)$. Since Z is a subspace of a metrizable space L, it is metrizable; hence $Z \in \mathscr{M}$. Since Y is an ANR for $\mathscr{M}$ there exists a neighborhood V of Z_0 in Z and a retraction $r : V \to Z_0$.

Now, if X is metrizable, A is a closed subset of X and $f : A \to Y$ is a mapping, then by Dugundji's extension theorem 2.3.6, the mapping

$$\Phi = \chi \circ f : A \to L$$

has an extension

$$\Psi : X \to L$$

such that $\Psi(X)$ is contained in the convex hull of $\Phi(A) \subset \chi(Y)$; hence $\Psi(X) \subset Z$ since Z is the convex hull of $\chi(Y)$. Then $U = \Psi^{-1}(V)$ is a neighborhood of A in X:

Clearly U is open since Ψ is continuous. Furthermore,

$$\Psi(A) = \chi\left(f(A)\right) \subset \chi(Y) = Z_0 \subset V$$

and thus it follows that

$$A \subset \Psi^{-1}(V) = U.$$

Now define

$$g : U \to Y$$

by

$$g(x) = \chi^{-1}(r(\Psi(x)))\ \forall x \in U.$$

Then g is an extension of f over U:

$$\begin{array}{lll}
a \in A \Rightarrow & & \\
g(a) & = \chi^{-1}(r(\Psi(a))) = \chi^{-1}(r(\Phi(a))) & \text{since } \Psi \text{ is an extension of } \Phi \\
 & = \chi^{-1}(r(\chi(f(a)))) & \\
 & = \chi^{-1}(\chi(f(a))) & \text{since } \chi(f(a)) \in \chi(Y) = Z_0 \\
 & & \text{and } r \text{ is a retraction} \\
 & = f(a). &
\end{array}$$

Hence Y is an ANE for $\mathscr{M}$. $\square$

2.6 Dominating spaces

The main goal of this section is to prove that any ANR is dominated by some simplicial polytope with the Whitehead topology. In order to prove that result, there are some definitions and lemmas to be studied. Reference: [5].

Lemma 2.6.1. *Let α be an open covering of an open subset W of a convex set Z in a locally convex space L (that is, the elements of α are contained in and open* in *W). Then α has an open refinement*

$$\gamma = \{W_\mu : \mu \in M\}$$

such that W_μ is convex for all $\mu \in M$.

Proof. Let $a \in W$; then there exists a neighborhood $U \in \alpha$ (that is, U is open *in* W) such that $a \in U$. Because U is open in W, and W is open in Z, it follows that U is open in Z. Now, because $Z \subset L$ has the relative topology, there exists an open set V in L such that

$$U = V \cap Z.$$

Now, since L is locally convex there exists a convex open set V' in L s.t.

$$a \in V' \subset V.$$

Now,

$$a \in V' \cap Z \subset V \cap Z = U,$$

and $V' \cap Z$ is convex since both V' and Z are convex. Furthermore, $V' \cap Z$ is open in W because it is open in Z. Hence $V' \cap Z$ is a convex open neighborhood of a in W, which we may denote W_a. Construct such a neighborhood W_a for each point $a \in W$; now

$$\gamma = \{W_a : a \in W\}$$

is an open cover of W whose elements are convex and for each W_a there exists a neighborhood $U \in \alpha$ such that $W_a \subset U$.

It follows that γ is a refinement of α as desired.

□

Lemma 2.6.2. *A normed space X (and in particular a Banach space such as $L = C(Y)$ from section (2.4)) is locally convex.*

Proof. A normed space is metrizable and hence, if d is a metric defining the topology in X (induced by the norm) then for every $a \in X$ and every neighborhood U of a in X there exists a real number $\delta > 0$ such that $B(a, \delta) \subset U$, and thus X is locally convex, since the balls $B(a, \delta)$ are convex. □

Lemma 2.6.3. *A convex subset A of a locally convex set X (in particular, a Banach space) is locally convex.*

Proof. Let $a \in A$, and let V be a neighborhood of a in A. Then there exists a neighborhood U of a in X such that $V = A \cap U$. Since X is locally convex, there exists a convex neighborhood U' of a in X such that $U' \subset U$. Since A is convex, the set $V' = A \cap U'$ is a convex neighborhood of a in A, and moreover, $V' = A \cap U' \subset A \cap U = V$. Hence A is locally convex. □

Definition 2.6.4 (Near maps). *Let $\alpha = \{U_\lambda : \lambda \in \Lambda\}$ be a covering of a topological space Y. Two maps $f, g : X \to Y$ are α-near if and only if*

$$\forall\, x \in X \ \exists\, \lambda \in \Lambda \text{ such that } f(x) \in U_\lambda \text{ and } g(x) \in U_\lambda.$$

Lemma 2.6.5. *If Y is a metrizable ANR then there exists an open covering α of Y such that any two α-near maps $f, g : X \to Y$ defined on an arbitrary space X are homotopic.*

Proof. By Theorem (2.4.3), we may consider Y as a closed subset of the convex set Z (= convex hull of $\chi(Y)$) in the Banach space $L = C(Y)$ (here we identify Y with its isometric image $\chi(Y) \subset Z$).

Since Y is an ANR there exists a neighborhood W of Y in Z and a retraction $r : W \to Y$.

Let β be some open covering of W (that is, its elements are open subsets of W). W is an open subset of a convex set Z in the locally convex set L (Lemma 2.6.2), and by Lemma (2.6.1), β has an open refinement

$$\gamma = \{W_\mu : \mu \in M\}$$

such that W_μ is convex for all $\mu \in M$. For each $\mu \in M$, denote $V_\mu = W_\mu \cap Y$. Then

$$\alpha = \{V_\mu : \mu \in M\}$$

is an open covering (open in Y) of Y.

Now let $f, g : X \to Y$ be two α-near maps defined on a space X. Since Z is convex, we can define a homotopy $k_t : X \to Z$ (where $0 \le t \le 1$) by setting

$$k_t(x) = (1-t)f(x) + tg(x) \ \forall \ x \in X, \ t \in I.$$

Claim: $k_t(x) \in W$ for all $x \in X$, $t \in I$
Proof: Let $x \in X$. Since f and g are α-near there exists $\mu \in M$ such that $f(x), g(x) \in V_\mu \subset W_\mu$. Since W_μ is convex,

$$k_t(x) \in W_\mu \subset W \ \forall \ t \in I. \square$$

Finally, define a homotopy $h_t : X \to Y$, where $0 \leq t \leq 1$, by setting

$$h_t(x) = r[k_t(x)] \ \forall \ x \in X, \ t \in I$$

Because both r and k_t are continuous, h_t is continuous, and moreover, since r is a retraction, we have that $h_0 = r \circ k_0 = r \circ f = f$ and, similarly, $h_1 = g$.

It follows that h_t is a homotopy from f to g.

$\square$

Definition 2.6.6 (Partial realizations of polytopes). *Let Y be a topological space, and let $\alpha = \{U_\lambda : \lambda \in \Lambda\}$ be a covering of Y. Let K be a simplicial polytope with the Whitehead topology, and let L be a subpolytope of K which contains all the vertices of K.*

A partial realization *of K in Y relative to α defined on L is a map*

$$f : L \to Y$$

such that for every closed simplex σ of K there exists $\lambda \in \Lambda$ for which

$$f(L \cap \sigma) \subset U_\lambda.$$

In the case where $L = K$ the function f is called a full realization *of K in Y relative to α.*

Lemma 2.6.7. *If a metrizable space Y is an ANR, then every open covering α of Y has an open refinement β such that every partial realization of any simplicial polytope K with Whitehead topology in Y relative to β extends to a full realization of K relative to α.*

Proof. Again, by Lemma (2.4.3), we may consider Y as a closed subspace of a convex set Z in the Banach space $C(Y)$.

Since Y is an ANR there exists an open neighborhood W of Y in Z and a retraction $r : W \to Y$. Being a convex set in a Banach space, Z is locally convex, by Lemma (2.6.3).

Let $\alpha = \{U_\lambda : \lambda \in \Lambda\}$ be a given open covering of Y; we construct a refinement β in the following way:

Let $y \in Y$ be any point, and choose a $\lambda \in \Lambda$ such that $y \in U_\lambda$. Since Z is locally convex there exists a convex neighborhood N_y of y in Z such that $N_y \subset W$ and $r(N_y) \subset U_\lambda$.

Set $V_y = N_y \cap Y$. It follows that

$$V_y = r(N_y \cap Y) \subset r(N_y) \subset U_\lambda$$

and thus

$$\beta = \{V_y : y \in Y\}$$

is an open refinement of α.

We now wish to show that β satisfies the given condition.

Let $f : L \to Y$ be a partial realization of a polytope K in Y relative to β. If $i : Y \hookrightarrow Z$ is the inclusion map, then consider the composition

$$\Phi = i \circ f : L \to Z.$$

We will construct an extension $\Psi : K \to Z$:

Let σ be any closed simplex of K, and let H_σ denote the convex hull of $\Phi(L \cap \sigma)$ in Z. Define

$$\overline{K}^n = K^n \cup L$$

where K^n is the n-skeleton of K. By induction we construct a sequence of maps

$$\Psi_n : \overline{K}^n \to Z \qquad n = 0, 1, 2, ...$$

satisfying the following conditions:

i) $\Psi_0 = \Phi$

ii) $\Psi_n | \overline{K}^{n-1} = \Psi_{n-1} \qquad n > 0$

iii) $\Psi_n(\overline{K}^n \cap \sigma) \subset H_\sigma \qquad$ for each closed simplex σ of K.

Now Ψ_0 is defined by i) - hence we assume that $n > 0$ and that Ψ_{n-1} has been constructed. We will extend Ψ_{n-1} to the interior of each n-dimensional simplex σ of K which is not contained in L. The boundary $\partial\sigma$ is a subset of

$\overline{K}^{n-1}$, and hence Ψ_{n-1} is defined there. Furthermore, because H_σ is a convex subset of a Banach space, it follows from Corollary (2.3.7) that $\Psi_{n-1}|_{\partial\sigma}$ has an extension

$$\kappa_\sigma : \sigma \rightarrow H_\sigma$$

We now define Ψ_n by setting

$$\Psi_n(x) = \begin{cases} \Psi_{n-1}(x) & x \in \overline{K}^{n-1} \\ \kappa_\sigma(x) & x \in \sigma \subset \overline{K}^n \setminus L \end{cases}$$

Claim: Ψ_n is continuous and satisfies the conditions i) - iii).

Proof: By induction on n. $\overline{K}^0 = L$ so Ψ_0 is continuous. Assume that Ψ_m is continuous for all $m < n$, and let σ be a simplex in $\overline{K}^n$. If σ is not a simplex of L, then by the definition of κ_σ, Ψ_n is continuous on σ.

If σ is a simplex of L then $\sigma \subset \overline{K}^m$ $\forall$ m and

$$\Psi_n|_\sigma = \Psi_{n-1}|_\sigma = ... = \Psi_0|_\sigma = \Phi|_\sigma$$

which is continuous. Hence Ψ_n is continuous on each simplex σ in $\overline{K}^n$ and so it is continuous on $\overline{K}^n$.

It is now easy to verify that Ψ_n satisfies the conditions *i) - iii)*. $\square$

Hence we have constructed a sequence of maps $\{\Psi_n : n = 0, 1, 2, ...\}$.

Next, we define a map $\Psi : K \rightarrow Z$ by setting

$$\Psi(x) = \Psi_n(x) \qquad \text{if } x \in \overline{K}^n.$$

Claim: Ψ is continuous

Proof: Let σ be a simplex in K, and let $n = \dim(\sigma)$. Then $\Psi|_{|\sigma|} = \Psi_n|_{|\sigma|}$ is continuous in $|\sigma|$ and hence Ψ is continuous on K. $\square$

We now wish to show that $\Psi(K) \subset W$. Let σ be any closed simplex of K. Because $f : L \rightarrow Y$ is a partial realization relative to β, there exists a point $y \in Y$ such that

$$\Phi(L \cap \sigma) = f(L \cap \sigma) \subset V_y \subset N_y$$

and since N_y is convex, we have that

$$\Psi(\sigma) \subset H_\sigma \subset N_y \subset W$$

Hence $\Psi(K) \subset W$.

Finally, we construct a map $g : K \rightarrow Y$ by

$$g(x) = r[\Psi(x)] \qquad \text{for every} \quad x \in K.$$

Because Ψ is an extension of Φ, we have that g is an extension of f. We just need to show that g is a full realization relative to α.

Let σ be any closed simplex in K. We have just shown that there is a $y \in Y$ such that $\Psi(\sigma) \subset N_y$, and from the way N_y is constructed we know that there is a $\lambda \in \Lambda$ such that $r(N_y) \subset U_\lambda$.

Hence

$$g(\sigma) = r[\Psi(\sigma)] \subset r(N_y) \subset U_\lambda \in \alpha$$

and so g *is* a full realization of K in Y relative to α.

□

Definition 2.6.8 (Dominating spaces). *A space X* dominates *the space Y if and only if there are maps*

$$\Phi : X \to Y$$
$$\Psi : Y \to X$$

such that the map $\Phi \circ \Psi : Y \to Y$ is homotopic to Id_Y. Then X is said to be a dominating space of Y.

Theorem 2.6.9. *Let Y be a metrizable ANR. Then there exists a simplicial polytope X with the Whitehead topology which dominates Y.*

Proof. By Lemma (2.6.5) there exists an open covering α of Y such that any two α-near maps $f, g : X \to Y$ for any space X are homotopic. By Lemma (2.6.7) there is an open refinement β of α such that any partial realization of any simplicial polytope K with the Whitehead topology in Y relative to β can be extended to a full realization of K in Y relative to α.

Being a metrizable space, Y is paracompact (Theorem 1.5.13) and fully normal (Proposition 1.6.4), and hence β has a neighborhood-finite open star refinement γ. Let X denote the geometric nerve of γ with the Whitehead topology. We will show that X dominates Y.

Let X^0 denote the 0-skeleton (i.e. the polytope corresponding to the simplicial complex consisting of the vertices of X) of X, and define a map

$$\Phi_0 : X^0 \to Y$$

in the following way:

For each open set $U \in \gamma$ pick a point $y_U \in U$, and define Φ_0 by setting

$$\Phi_0(v_U) = y_U$$

where v_U is the vertex of X corresponding to U. Now, since X^0 is discrete, Φ_0 is continuous. Furthermore, if $|\sigma|$ is a closed simplex in X, whose vertices are $v_{U_0}, ..., v_{U_q}$, then by the definition of a nerve,

$$U_0 \cap U_1 \cap ... \cap U_q \neq \emptyset$$

Now, because γ is a star refinement of β, there exists an open set $V_\sigma \in \beta$ which contains each $U_i, \quad i = 0, 1, ..., q$. It follows that $\Phi_0(|\sigma| \cap X^0) \subset V_\sigma$, and so Φ_0 is a partial realization of X in Y relative to β defined on X^0.

By the choice of the covering β, Φ_0 extends to a full realization

$$\Phi : X \to Y$$

relative to α. Then for any closed simplex $|\sigma|$ of X there exists an open set $W_\sigma \in \alpha$ such that

$$\Phi(|\sigma|) \subset W_\sigma$$

for each closed simplex $|\sigma|$ of X. Considering the proof of Lemma (2.6.7) we may assume that $V_\sigma \subset W_\sigma$ for each simplex σ of X.

Now consider the canonical map

$$\kappa : Y \to X$$

of the locally finite covering γ as defined in the proof of Lemma (2.3.4). We will show that the maps

$$\Phi \circ \kappa : Y \to Y$$

and

$$Id : Y \to Y$$

are α-near, so that by the definition of the covering α they will be homotopic.

Let $y \in Y$ be an arbitrarily chosen point, and let $U_0, U_1, ...U_q$ be the sets of β containing y. Then $\kappa(y)$ is a point of the open simplex $\langle\sigma\rangle$ of X with vertices $v_{U_0}, ..., v_{U_q}$. It follows that

$$\Phi(\kappa(y)) \in W_\sigma$$

On the other hand,

$$y \in U_i \subset V_\sigma \subset W_\sigma$$

and so $\Phi \circ \kappa$ and Id_Y are α-near.

Hence $\Phi \circ \kappa$ is homotopic to Id_Y and so X dominates Y. $\square$

2.7 Manifolds and local ANRs

References: [3], [5]

In this section we will show that each topological manifold is indeed an ANR. In this section, ANR denotes an ANR for the class $\mathscr{M}$ of metrizable spaces.

Definition 2.7.1 (Topological manifold). *A topological space X is called a* topological n-manifold*, where $n \in \mathbb{N}$, if*

i) X is Hausdorff

ii) X is N_2

iii) Each point $x \in X$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$.

Remark 2.7.2. *Recall from Topology II that a topological manifold is metrizable and separable.*

Definition 2.7.3 (Local ANR/ANE). *A metrizable space Y is a* local ANR *if each point $y \in Y$ has a neighborhood which is an ANR.*

For any class of spaces there is a similar definition if ANR is replaced by ANE, but in the case of metric spaces these two concepts are the same by Theorem (2.5.1).

Example 2.7.4. *A topological manifold is a local ANR.*

Proof. Let Y be a topological n-manifold, and let $y \in Y$ be any point. Now y has a neighborhood V which is homeomorphic to $\mathbb{R}^n$. Let X be any metrizable space and let A be any closed subset of X. Let

$$f : A \to V$$

be any continuous function, and let

$$h : V \to \mathbb{R}^n$$

be the homeomorphism between V and $\mathbb{R}^n$. Now

$$h \circ f : A \to \mathbb{R}^n$$

is a mapping, and furthermore, $\mathbb{R}^n$ is a locally convex topological linear space. Hence by Dugundji's extension theorem (2.3.6) there is a continuous extension

$$g : X \to \mathbb{R}^n,$$

and hence

$$h^{-1} \circ g : X \to V$$

is a continuous extension of f. It follows that V is an ANE (in fact an AE, and thus an ANE); hence, since it is also metrizable it is an ANR, and so Y is a local ANR. □

Lemma 2.7.5. *Every open subspace of an ANE for the class $\mathscr{C}$ is an ANE for $\mathscr{C}$.*

Proof. Let Y be an ANE for the class $\mathscr{C}$ and let W be an open subspace of Y. Let $f : A \to W$ be any map defined on a closed subspace A of an arbitrary space X from $\mathscr{C}$. Since Y is an ANE for $\mathscr{C}$, the composed mapping

$$i \circ f : A \to W \hookrightarrow Y$$

has an extension

$$g : V \to Y$$

over a neighborhood V of A in X. Now denote

$$U = g^{-1}(W) \subset V;$$

since g is continuous and W is open in Y, U is an open subset of V and hence of X. Furthermore, since $g(A) = f(A) \subset W$, U is a neighborhood of A in X.

Denote

$$h = g|U : U \to W,$$

now h is an extension of f over U and hence W is an ANE for $\mathscr{C}$.

□

Theorem 2.7.6. *A separable metrizable space X which is a local ANR is an ANR.*

Proof. The theorem will be proved in three steps:

i) If X is the union of two open ANRs it is an ANR.

ii) If X is the union of countably many disjoint open ANRs it is an ANR.

iii) If X is the union of arbitrarily many open ANRs it is an ANR.

The proof goes as follows:

i) Assume that $X = A_1 \cup A_2$ where A_1 and A_2 are open ANRs, or equivalently, ANEs. Let $f : B \to X$ where B is a closed subspace of some metrizable space Y. Now the sets

$$F_1 = B \setminus f^{-1}(A_2); \qquad F_2 = B \setminus f^{-1}(A_1)$$

are disjoint and closed in B; hence also in Y. Now since Y is normal there are disjoint open sets Y_1 and Y_2 in Y such that

$$F_1 \subset Y_1; \qquad F_2 \subset Y_2.$$

Now $Y_0 = Y \setminus (Y_1 \cup Y_2)$ is closed in Y. Denote $B_i = Y_i \cap B$ where $i = 0, 1, 2$. We then see that

$$a)\ f(B_0) \subset A_1 \cap A_2; \quad b)\ f(B_1) \subset A_1; \quad c)\ f(B_2) \subset A_2$$

since

a)

$$\begin{aligned} x \in B_0 &\Rightarrow x \in Y_0 \wedge x \in B \\ &\Rightarrow x \notin (Y_1 \cup Y_2) \wedge x \in B \\ &\Rightarrow x \notin (F_1 \cup F_2) \wedge x \in B \\ &\Rightarrow x \in f^{-1}(A_2) \wedge x \in f^{-1}(A_1) \\ &\Rightarrow f(x) \in A_2 \wedge f(x) \in A_1. \end{aligned}$$

b)

$$\begin{aligned} x \in B_1 \Rightarrow x \in Y_1 \wedge x \in B& \\ &\Rightarrow x \notin Y_2 \wedge x \in B \\ &\Rightarrow x \notin F_2 \wedge x \in B \\ &\Rightarrow x \in f^{-1}(A_1) \\ &\Rightarrow f(x) \in A_1. \end{aligned}$$

c) In the same way $f(B_2) \subset A_2$.

Because $B_0 \subset Y_0$ is closed and $A_1 \cap A_2$ is an ANR by Lemma (2.7.5), there is an extension of the mapping $f|_{B_0} : B_0 \to A_1 \cap A_2$ to some open neighborhood U_0 of B_0 in Y_0. This extension and the original mapping f agree on $B_0 = B \cap U_0$, so combined they define a function $g : B \cup U_0 \to X$. Because

$$U_0 = (U_0 \cup B) \cap Y_0,$$

U_0 is closed in $U_0 \cup B$. Since B is closed in Y it is also closed in $U_0 \cup B$. Hence, by Lemma (1.4.1), g is continuous.

Now it holds that

$$g(U_0 \cup B_1) \subset A_1; \quad g(U_0 \cup B_2) \subset A_2$$

since

$$g(U_0 \cup B_1) = g(U_0) \cup f(B_1) \subset A_1; \quad g(U_0 \cup B_2) = g(U_0) \cup f(B_2) \subset A_2$$

and $Y_0 \setminus U_0$ is closed in Y.

Also, the set $U_0 \cup B_1$ is closed in $U_0 \cup Y_1$ because

$$(U_0 \cup Y_1) \setminus (U_0 \cup B_1) = Y_1 \setminus B_1 = Y_1 \setminus B$$

which is open in Y. Now, since A_1 is an ANR we may extend $g| : (U_0 \cup B_1) \to A_1$ to a mapping $g_1 : U_1 \to A_1$ where U_1 is an open neighborhood of $U_0 \cup B_1$ in $U_0 \cup Y_1$. Since $Y_0 \setminus U_0$ was closed in Y, the set $U_0 \cup Y_1$ is open in $Y_0 \cup Y_1$, and hence U_1 is open in $Y_0 \cup Y_1$.

Analogously, one may extend the mapping $g| : (U_0 \cup B_2) \to A_2$ to an open neighborhood U_2 of $U_0 \cup B_2$ in $U_0 \cup Y_2$, and again we see that U_2 is open in $Y_0 \cup Y_2$.

Denote $U = U_1 \cup U_2$ and define $F : U \to X$ by setting

$$F(u) = \begin{cases} g_1(u) & \text{if } u \in U_1, \\ g_2(u) & \text{if } u \in U_2. \end{cases}$$

If now $x \in U_0 = U_1 \cap U_2$ then $g(x) = g_1(x) = g_2(x)$. It follows that F is well defined. Because $U_1 = U \setminus Y_2$ and $U_2 = U \setminus Y_1$ the sets U_1 and U_2 are closed in U; hence by Lemma (1.4.1) F is continuous. It is clear that F is an extension of f and it remains to show that U is a neighborhood of B in Y.

We have already seen that U_i is open in $Y_0 \cup Y_i$ for $i = 1, 2$, and it follows that

$$Y \setminus U = ((Y_0 \cup Y_1) \setminus U_1) \cup ((Y_0 \cup Y_2) \setminus U_2)$$

which is then closed. Hence U is open. It follows that X is an ANE, or equivalently, an ANR.

ii) Now assume that $X = \bigcup_{n \in \mathbb{N}} A_n$ where the A_n are disjoint open ANRs. Then suppose that X is embedded as a closed subset of some metrizable space Z, and let d be a metric on Z. Now, since each A_i is the complement of an open subset of X it is closed in X and hence in Z. Find some collection $\{U_n : n \in \mathbb{N}\}$ of disjoint open subsets of Z such that $A_n \subset U_n$ for each $n \in \mathbb{N}$. (This may be done, for instance, by choosing $U_n = \{z \in Z : d(z, A_n) < d(z, X \setminus A_n)\}$.) A_n being an ANR, and also being a closed subset of U_n, there is some open set $V_n \subset U_n$ and a retraction $r_n : U_n \to A_n$. These retractions then define a retraction $r : \bigcup_{n \in \mathbb{N}} V_n \to X$ by setting $r(x) = r_n(x)$ whenever $x \in V_n$. Because $\bigcup_{n \in \mathbb{N}} V_n$ is an open subset of Z which contains X, we see that X is an ANR.

iii) We now assume that X is any union of open ANRs A_i. Since X is metrizable and separable it is Lindelöf, and so there exists a countable set $\{A_i : i \in \mathbb{N}\}$ of these ANRs such that $X = \bigcup_{i\in\mathbb{N}} A_i$. However, the A_i are not necessarily disjoint so we cannot yet use part ii).

Define new open sets

$$U_n = \bigcup_{i=1}^{n} A_i.$$

By part i) we see that each U_n is an ANR, and it is clear that $X = \bigcup_{n\in\mathbb{N}} U_n$ and that $U_n \subset U_{n+1}$ for all $n \in \mathbb{N}$.

Define open sets V_n for all $n \in \mathbb{N}$ by setting

$$V_n = \{x \in X : d(x, U_n^C) < \frac{1}{n}\}$$

where d is a metric defining the topology on X. Now $V_n \subset U_n$ and V_n is open; hence V_n is an ANR. Furthermore,

$$X = \bigcup_{n\in\mathbb{N}} V_n; \quad \text{and } \forall n \in \mathbb{N} : \overline{V}_n \subset V_{n+1}$$

Now define open sets W_n for all $n \in \mathbb{N}$ by setting

$$W_1 = V_1; \quad W_2 = V_2; \quad W_n = V_n \setminus \overline{V}_{n-2} \text{ when } n \geq 3.$$

It is clear that each W_n is open in X and that $W_n \subset V_n$. Hence W_n is an ANR. Furthermore, $V_n \setminus V_{n-1} \subset W_n$ and so

$$X = \bigcup_{n\in\mathbb{N}} W_n = \left(\bigcup_{n\in\mathbb{N}} W_{2n}\right) \cup \left(\bigcup_{n\in\mathbb{N}} W_{2n-1}\right)$$

where $\bigcup_{n\in\mathbb{N}} W_{2n}$ and $\bigcup_{n\in\mathbb{N}} W_{2n-1}$ are unions of disjoint open ANRs; hence by ii) they are ANRs, and so by i) X is an ANR.

$\square$

Theorem 2.7.7. *A topological manifold is an ANR.*

Proof. By Example (2.7.4) a manifold is a local ANR, and by Theorem (2.7.6) it is an ANR.

□

Chapter 3

Homotopy theory

In order to show that an ANR is homotopy equivalent to a CW-complex we will use elements of homotopy theory such as higher homotopy groups and weak homotopy equivalence. This chapter provides the required machinery in addition to a survey of CW-complexes, and concludes with the result that any ANR is homotopy equivalent to a CW-complex.

3.1 Higher homotopy groups

References: [7], [8]

Just like the elements of the fundamental group $\pi_1(X, x_0)$ of a topological space X based at a point $x_0 \in X$ can be seen as equivalence classes of mappings $f : S^1 \to X$, we may define new groups $\pi_n(X, x_0)$ whose elements are equivalence classes of mappings $g : S^n \to X$. Such a group will be called the n^{th} homotopy group of X.

In this section we define the higher homotopy groups of a space and then prove some of their basic properties.

We begin by defining the n^{th} homotopy group as a set only; in the set of mappings $(S^n, e_0) \to (X, x_0)$ where $e_0 = (0, 0, ..., 0, 1) \in S^n$ and x_0 are the base points of the spaces S^n and X, two mappings $f, g : (S^n, e_0) \to (X, x_0)$ belong to the same equivalence class if and only if $f \simeq g$ rel e_0. In that case the equivalence class is written $[f] = [g]$. We define $\pi_n(X, x_0) = \{[f] | f : (S^n, e_0) \to (X, x_0)\}$.

The proofs of the following basic lemmas etc were covered in the course Homotopy theory, so here I will only state the results.

From now on we will denote by $[Y, B; X, x_0]$ the set of homotopy classes rel B of mappings $(Y, B) \to (X, x_0)$.

Lemma 3.1.1. *Define the mapping $\Phi : \overline{B}^n \to S^n$ by setting*

$$\Phi(x) = (2\sqrt{1 - |x|^2}x, 2|x|^2 - 1) \in \mathbb{R}^n \times \mathbb{R}.$$

Then $\Phi(S^{n-1}) = e_0 = (0, 0, ..., 0, 1) \in \mathbb{R}^{n+1}$ and Φ induces a homeomorphism $\Phi_\sharp : \overline{B}^n / S^{n-1} \to S^n$.

It follows from this that the mapping $[f] \mapsto [f \circ \Phi]$ is a bijection $\pi_n(X, x_0) \to [\overline{B}^n, S^{n-1}; X, x_0]$.

Since $\overline{B}^n$ is homeomorphic to I^n and the homeomorphism between them takes the boundary of one to the boundary of the other, we have in fact a bijection between the set $\pi_n(X, x_0)$ as defined and the set $[I^n, \partial I^n; X, x_0]$. Hence the elements of $\pi_n(X, x_0)$ may be seen as equivalence classes rel ∂I^n of mappings $f : (I^n, \partial I^n) \to (X, x_0)$.

We define a binary operation in $\pi_n(X, x_0)$ by setting $[f][g] = [fg]$, where the product fg is defined analogously to as in the case of the fundamental group:

For two maps $f, g : (I^n, \partial I^n) \to (X, x_0)$ their product is

$$fg : (I^n, \partial I^n) \to (X, x_0),$$

$$fg\left((x_1, x_2, ..., x_n)\right) = \begin{cases} f\left((2x_1, x_2, ..., x_n)\right), & 0 \le x_1 \le \frac{1}{2} \\ g\left((2x_1 - 1, x_2, ..., x_n)\right), & \frac{1}{2} \le x_1 \le 1 \end{cases}$$

Then, since fg is continuous on the two closed sets $\{(x_1, ..., x_n) \in I^n : x_1 \le \frac{1}{2}\}$ and $\{(x_1, ..., x_n) \in I^n : x_1 \ge \frac{1}{2}\}$ and agrees on their intersection it is continuous on all of I^n by Lemma (1.4.1), and clearly $fg(\partial I^n) = x_0$, so fg is well defined.

Lemma 3.1.2. *The set $\pi_n(X, x_0)$ with the binary operation defined above is a group.*

Let (X, x_0) and (Y, y_0) be topological spaces with base points x_0 and y_0, and let $f : (X, x_0) \to (Y, y_0)$ be a mapping. If $g : (S^n, s) \to (X, x_0)$ is a mapping then so is $f \circ g : (S^n, s) \to (Y, y_0)$; and furthermore, if g_1, g_2 :

$(S^n, s) \to (X, x_0)$ such that $g_1 \simeq g_2$ rel s, then $f \circ g_1 \simeq f \circ g_2$ rel s. Hence we obtain a well-defined mapping

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

induced by f which is defined by $f_*([g]) = [f \circ g]$ whenever $g \in \pi_n(X, x_0)$.

Proposition 3.1.3. *i) f_* is a homomorphism.*

ii) $(Id_X)_* = Id_{\pi_n(X,x_0)}$.

iii) $(f \circ g)_* = f_* \circ g_*$.

iv) If $f_0 \simeq f_1$ rel x_0 then $(f_0)_* = (f_1)_*$.

v) If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is a group isomorphism for all $n \in \mathbb{N}$ and for all base points $x_0 \in X$.

Definition 3.1.4 (Relative homotopy group). *Let (X, A) be a pair of topological spaces - that is, X is a topological space and A is a subspace of X. Now we define the relative homotopy group of the space X with respect to A at a base point $x_0 \in A$ as follows:*

Given the n-cube I^n, where $n \geq 1$, let I^{n-1} denote the face of I^n where the coordinate $t_n = 0$. The union of the remaining faces will be denoted J^{n-1}. Then we have

$$\partial I^n = I^{n-1} \cup J^{n-1}; \qquad \partial I^{n-1} = I^{n-1} \cap J^{n-1}.$$

Consider mappings

$$f : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0);$$

that is, continuous functions $f : I^n \to X$ such that $f(I^{n-1}) \subset A$ and $f(J^{n-1}) = \{x_0\}$.

We denote by $\pi_n(X, A, x_0)$ the set of homotopy classes $[f]$ rel J^{n-1} of such mappings f (where the homotopy maps I^{n-1} into A), and we define multiplication in $\pi_n(X, A, x_0)$ as we did earlier in $\pi_n(X, x_0)$. We may show in the exact same way that $\pi_n(X, A, x_0)$ with this multiplication is a group when $n \geq 2$, and we call it thus the n^{th} relative homotopy group of X with respect to A at x_0.

Proposition 3.1.5. *If $\alpha \in \pi_n(X, A, x_0)$ is represented by a mapping*

$$f : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$$

such that $f(I^n) \subset A$, then $\alpha = 0$.

Proof. Define a homotopy

$$F : I^n \times I \to X$$

by setting

$$F\left((t_1, ..., t_{n-1}, t_n), t\right) = f(t_1, ..., t_{n-1}, t + t_n - tt_n).$$

Then, since $t+t_n-tt_n = t(1-t_n)+t_n \in I$ we get that $F_t : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ for all $t \in I$ and

$$F_0 = f; \qquad F_1(I^n) = \{x_0\}$$

since $(t_1, ..., t_{n-1}, 1) \in J^{n-1}$ and $f(J^{n-1}) = \{x_o\}$. Hence $\alpha = 0$. □

3.2 The exact homotopy sequence of a pair of spaces

Reference: [7]

Let X be a topological space, let $x_0 \in A \subset X$ and let $n \geq 1$. We will define a function

$$\delta : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0).$$

which is a group homomorphism for $n > 1$. Assume that $\alpha \in \pi_n(X, A, x_0)$. Then α is the equivalence class of some mapping $f : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$.

If $n = 1$, then I^{n-1} is a point; hence $f(I^{n-1})$ is a point of A defining a path component $\beta \in \pi_{n-1}(A, x_0)$ of A.

If $n > 1$, then the restriction $f|_{I^{n-1}}$ is a mapping $(I^{n-1}, \partial I^{n-1}) \to (A, x_0)$ and thus it represents some element (that is, a homotopy class) $\beta \in \pi_{n-1}(A, x_0)$.

Obviously the element $\beta \in \pi_{n-1}(A, x_0)$ does not depend on the choice of f representing the homotopy class $\alpha \in \pi_n(X, A, x_0)$. Thus we may define the function δ by setting

$$\delta(\alpha) = \beta.$$

The function δ will be called the *boundary operator*.

The following propositions are trivial:

Proposition 3.2.1.

$$\delta(e_{\pi_n(X,A,x_0)}) = e_{\pi_{n-1}(A,x_0)}. \quad \square$$

Proposition 3.2.2. *If* $n > 1$ *then* δ *is a group homomorphism.* $\square$

The inclusion maps

$$i : (A, x_0) \hookrightarrow (X, x_0), \qquad j : (X, x_0) \hookrightarrow (X, A, x_0)$$

induce functions

$$i_* : \pi_n(A, x_0) \to \pi_n(X, x_0), \qquad j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0)$$

which are homomorphisms for $n \geq 1$ and $n \geq 2$, respectively.

(Note that j is in fact $(X, \{x_0\}, x_0) \hookrightarrow (X, A, x_0)$ but that since $f : (I^n, \partial I^n) \to (X, x_0)$ is equivalent to $f : (I^n, I^{n-1}, J^{n-1}) \to (X, \{x_0\}, x_0)$ we may identify $\pi_n(X, x_0)$ with $\pi_n(X, \{x_0\}, x_0)$.)

We may now define a sequence

$$... \xrightarrow{j_*} \pi_{n+1}(X, A, x_0) \xrightarrow{\delta} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\delta} \pi_{n-1}(A, x_0) \to ...$$

$$\to \pi_1(X, A, x_0) \xrightarrow{\delta} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0).$$

which is called the *homotopy sequence* of the pair (X, A) with respect to the base point $x_0 \in A$.

Definition 3.2.3 (Exact sequence). *A sequence*

$$... \to G_{i+1} \xrightarrow{f_i} G_i \xrightarrow{f_{i-1}} G_{i-1} \to ...$$

of groups and homomorphisms is said to be exact *if, for each* $i \in \mathbb{N}$, $Ker(f_{i-1}) = Im(f_i)$.

Remark 3.2.4. *If the sequence ends, as in the case of the homotopy sequence, then no restriction is put on the image of the last mapping. Furthermore, if the last sets are not groups and thus the last mappings are not homomorphism, the same definition of exatness holds even for those last steps.*

Theorem 3.2.5 (The exact homotopy sequence of a pair). *The homotopy sequence of any pair (X, A) with respect to any base point $x_0 \in A$ is exact.*

Proof. The proof breaks up into six statements:

i) $j_* i_* = 0$.

ii) $\delta j_* = 0$.

iii) $i_* \delta = 0$.

iv) If $\alpha \in \pi_n(X, x_0)$ and $j_*(\alpha) = 0$, then there exists an element $\beta \in \pi_n(A, x_0)$ such that $i_*(\beta) = \alpha$.

v) If $\alpha \in \pi_n(X, A, x_0)$ and $\delta(\alpha) = 0$, then there exists an element $\beta \in \pi_n(X, x_0)$ such that $j_*(\beta) = \alpha$.

vi) If $\alpha \in \pi_{n-1}(A, x_0)$ and $i_*(\alpha) = 0$, then there exists an element $\beta \in \pi_n(X, A, x_0)$ such that $\delta(\beta) = \alpha$.

Now, from i) we get that $Im(i_*) \subset Ker(j_*)$; and from iv) we get that $Ker(j_*) \subset Im(i_*)$; hence $Im(i_*) = Ker(j_*)$. Similarly, ii) and v) give $Im(j_*) = Ker(\delta)$, and iii) and vi) give $Im(\delta) = Ker(i_*)$. Thus it suffices to show that the statements i)-vi) are true.

Proof of i) For each $n > 0$, let $\alpha \in \pi_n(A, x_0)$ and let $f : (I^n, \partial I^n) \to (A, x_0)$ be a map belonging to the homotopy class α. Now the element $j_* i_*(\alpha) \in \pi_n(X, A, x_0)$ is the homotopy class of the map

$$j \circ i \circ f : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0),$$

and since $(j \circ i \circ f)(I^n) \subset A$ we get from Proposition (3.1.5) that $j_* i_* \alpha = 0$. Since α was an arbitrary element of $\pi_n(A, x_0)$, it follows that $j_* i_* = 0$.

Proof of ii) For each $n > 0$, let $\alpha \in \pi_n(X, x_0)$ and choose a mapping $f : (I^n, \partial I^n) \to (X, x_0)$ belonging to the homotopy class α. Then the element $\delta j_*(\alpha)$ is determined by the restriction $(j \circ f)|_{I^{n-1}} = f|_{I^{n-1}}$, and since $f(I^{n-1}) = \{x_0\}$ it follows that $(\delta \circ j_*)(\alpha) = 0$. Hence $\delta \circ j_* = 0$.

Proof of iii) For each $n > 0$, let $\alpha \in \pi_n(X, A, x_0)$ and choose a map $f : (I^n; I^{n-1}, J^{n-1}) \to (X, A, x_0)$ which is in the homotopy class α. Then the element $i_*\delta(\alpha) \in \pi_{n-1}(X, x_0)$ is determined by the restriction $g = f|_{I^{n-1}}$. Define a homotopy $G : I^{n-1} \times I \to X$ by setting

$$G\left((t_1, ..., t_{n-1}), t\right) = f(t_1, ..., t_{n-1}, t).$$

Then $G_0 = g$, $G_1(I^{n-1}) = \{x_0\}$ and $G_t : (I^{n-1}, I^{n-2}) \to (X, x_0)$ if $n > 1$, hence $[g] = 0 \in \pi_{n-1}(X, x_0)$. If $n = 1$ then $G(I^{n-1} \times I)$ is contained in one path component of X - the one containing x_0. Hence $[g] = 0 \in \pi_0(X, x_0)$. This implies that $(i_* \circ \delta)(\alpha) = 0$; hence $i_* \circ \delta = 0$.

Proof of iv) Let $f : (I^n, \partial I^n) \to (X, x_0)$ be a mapping in the homotopy class α. Then since $j_*(\alpha) = 0$, there must exist a homotopy $F : I^n \times I \to X$ such that $F_0 = f, F_1(I^n) = \{x_0\}$ and $F_t : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ for all $t \in I$. Define a new homotopy $G : I^n \times I \to X$ by setting

$$\begin{aligned} G\left((t_1, ..., t_{n-1}, t_n), t\right) &= F\left((t_1, ..., t_{n-1}, 0), 2t_n\right), \quad &\text{if } 0 \leq 2t_n \leq t, \\ &F\left((t_1, ..., t_{n-1}, \tfrac{2t_n - t}{2-t}), t\right), \quad &\text{if } t \leq 2t_n \leq 2. \end{aligned}$$

Then $G_0 = f$, $G_1(I^n) \subset A$ and $G_t(\partial I^n) = \{x_0\}$ for all $t \in I$. Now G_1 belongs to some homotopy class β of $\pi_n(A, x_0)$, and hence $i_*(\beta) = \alpha$.

Proof of v) First, let's assume that $n > 1$. Let $f : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ belong to the homotopy class α - then since $\delta(\alpha) = 0$ there is a homotopy $G : I^{n-1} \times I \to A$ such that $G_0 = f|_{I^{n-1}}$, $G_1(I^{n-1}) = \{x_0\}$ and $G_t(\partial I^{n-1}) = \{x_0\}$ for all $t \in I$. We will define a homotopy

$$H : \partial I^n \times I \to A$$

by setting

$$H(x, t) = \begin{cases} G(x, t) & \text{if } x \in I^{n-1} \\ x_0 & \text{if } x \in J^{n-1}. \end{cases}$$

Since $H_0 = f|_{\partial I^n}$, it follows from Corollary (3.3.15) that H has an extension $F : I^n \times I \to X$ such that $F_0 = f$. Since $F_1(\partial I^n) = H_1(\partial I^n) = x_0$, F_1 belongs to some homotopy class $\beta \in \pi_n(X, x_0)$, and since $F_t : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ for all $t \in I$ we get that $j_*(\beta) = \alpha$.

In the case $n = 1$, α is represented by a path $f : I \to X$ such that $f(0) \in A$ and $f(1) = x_0$. Here the condition $\delta(\alpha) = 0$ means that $f(0)$ is contained in the same path component of A as x_0. Thus there is a path $\gamma : I \to A$ such that $\gamma(0) = x_0$ and $\gamma(1) = f(0)$. We may now define a homotopy $F : I \times I \to X$ by setting

$$F(s,t) = \begin{cases} \gamma\left((1-t) + (1+t)s\right) & \text{when } 0 \leq s \leq \frac{t}{1+t} \\ f\left((1+t)s - t\right) & \text{when } \frac{t}{1+t} \leq s \leq 1 \end{cases}$$

such that $F_0 = f$, $F_t(0) \in \gamma(I) \subset A$, $F_t(1) = f(1) = x_0$ for all $t \in I$ and $F_1(0) = \gamma(0) = x_0$. Then F_1 belongs to a homotopy class $\beta \in \pi_1(X, x_0)$ and the homotopy F implies that $j_*(\beta) = \alpha$.

Proof of vi) Let's first assume that $n > 1$. Let $f : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$ belong to the homotopy class α; then from the assumption $i_*(\alpha) = 0$ we know that there must exist a homotopy $F : I^{n-1} \times I \to X$ such that $F_0 = f$, $F_1(I^{n-1}) = \{x_0\}$ and $F_t(\partial I^{n-1}) = \{x_0\}$ for all $t \in I$. Define a mapping $g : I^n \to X$ by setting

$$g(t_1, ..., t_{n-1}, t_n) = F\left((t_1, ..., t_{n-1}), t_n\right),$$

then $g : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ belongs to some homotopy class $\beta \in \pi_n(X, A, x_0)$, and since $g|_{I^{n-1}} = f$, we get $\delta(\beta) = \alpha$.

Now consider the case where $n = 1$; then α is a path component of A. The assumption $i_*(\alpha) = 0$ means that α is contained in the path component of X containing x_0. Let $f : I \to X$ be the constant path $f(t) = x_0$. This path represents a homotopy class $\beta \in \pi_1(X, A, x_0)$, and since $f(0) \in \alpha$, we get $\delta(\beta) = \alpha$.

□

3.3 Adjunction spaces and the method of adjoining cells

CW-complexes may be constructed through a method referred to as *adjoining cells*. Before defining what a CW-complex is we will take a closer look at the method we will use to build them. First of all we will define what an adjunction space is.

Let X and Y be topological spaces, and let A be a closed subset of X. Let $f : A \to Y$ be a continuous map. Denote by W the topological sum $X + Y$ - that is, W is the disjoint union of X and Y topologized such that a subset $V \subset W$ is open if and only if $V \cap X$ is open in X and $V \cap Y$ is open in Y.

We will define a relation $\sim$ on W in the following way: If u and v are two elements of W, then $u \sim v$ if and only if at least one of the following four equations makes sense and holds:

$$u = v, \quad f(u) = v, \quad u = f(v), \quad f(u) = f(v).$$

It is clear that $\sim$ is an equivalence relation.

Hence, by identifying the elements which are equivalent to each other, we "glue" the spaces X and Y together "along" the closed subset A.

Definition 3.3.1 (Adjunction space). *The quotient space $Z = W/\sim$ of the space W over the equivalence relation $\sim$ is the* adjunction space *obtained by adjoining X to Y by means of the given map $f : A \to Y$.*

Consider the canonical projection

$$p : W \to Z$$

Since p coinduces the topology on Z, it is of course continuous. Furthermore, we have:

Proposition 3.3.2. *The restriction*

$$i = p|Y : Y \to Z$$

is an embedding.

Proof. Since p is the restriction of a continuous map it is continuous. Assume that $u, v \in Y$ such that $p(u) = p(v)$, or in other words, $u \sim v$. Now, because u and v are elements of Y, the first of the four equations is the only one making sense and hence $u = v$. Now we know that i is injective, and it remains to show that it is a closed map.

Let C be a closed subset of Y. Now because p is an identification map, $p(C)$ is closed in Z if and only if $D = p^{-1}(p(C))$ is closed in W. From the definition of $\sim$ we get that $D = f^{-1}(C) \cup C$. Since C is closed in Y and f is continuous, we have that $f^{-1}(C) = D \cap X$ is closed in A and hence in X, and $C = D \cap Y$ was closed in Y, so D is closed in W. Hence i is a closed map.

□

If we let $C = Y$ in the proof above we get the following result:

Corollary 3.3.3. *The image $p(Y)$ is a closed subspace of the adjunction space Z.*

From now on we may thus identify Y with $p(Y)$ and view Y as a closed subspace of the adjunction space Z.

Proposition 3.3.4. *The restriction*

$$j = p|(X \setminus A) : (X \setminus A) \to Z$$

is an embedding.

Proof. The injectivity part is proved as in Proposition (3.3.2), and since p is continuous then so is j. We proceed to show that j is an open map.

Let U be an open subset of $X \setminus A$. Again because p is an identification map, $p(U)$ is open in Z if and only if $V = p^{-1}(p(U))$ is open in W. By the definition of $\sim$ we get that $U = V$ and since $X \setminus A$ is open in W we have that V is also open in W. Hence j is an open map.

□

By setting $U = X \setminus A$ in the proof above we obtain:

Corollary 3.3.5. *The image $p(X \setminus A)$ is an open subspace of the adjunction space Z.*

We may now identify $X \setminus A$ with its image under p and view $X \setminus A$ as an open subspace of the adjunction space Z.

It is also obvious that $p(Y)$ and $p(X \setminus A)$ are disjoint and hence

Proposition 3.3.6. $Z = p(Y) \cup p(X \setminus A)$

We may thus consider Z to be the disjoint union of Y and $X \setminus A$ glued together by a topology defined using the mapping f. Next we will see that the separation properties T_1 and normality are conserved in adjunction spaces.

Proposition 3.3.7. *Assuming that the spaces X and Y are*

i) T_1*,*

ii) normal,

then so is the adjunction space Z.

Proof. i) Let $z \in Z$. If $z \in Y$ then $\{z\}$ is closed in Y since Y is T_1. Since Y is closed in Z, then $\{z\}$ is closed also in Z.

If $z \notin Y$ then $p^{-1}(z)$ is one single point in $X \setminus A$ and it is closed in X since X is T_1. But X is closed in W and so $p^{-1}(z)$ is closed in W, and so $\{z\}$ is closed in Z since p is an identification map.

ii) Let F_1 and F_2 be two disjoint, closed subsets of Z. Then $F_1 \cap Y$ and $F_2 \cap Y$ are disjoint closed subsets of Y and so since Y is normal, there exist disjoint open neighborhoods U_1 and U_2 of $F_1 \cap Y$ and $F_2 \cap Y$ in Y, such that $\overline{U_1} \cap \overline{U_2} = \emptyset$. Since Y is closed in Z, these are also their closures in Z.

Now define

$$K_1 = F_1 \cup \overline{U_1}, \qquad K_2 = F_2 \cup \overline{U_2},$$

which are disjoint closed subsets of Z. Then the sets

$$J_1 = p^{-1}(K_1) \cap X, \qquad J_2 = p^{-1}(K_2) \cap X$$

are disjoint and closed in X. Because X is normal, there are disjoint open neighborhoods V_1 and V_2 of J_1 and J_2.

Consider the subsets

$$G_1 = p(V_1 \setminus A) \cup U_1, \qquad G_2 = p(V_2 \setminus A) \cup U_2$$

of Z. We wish to show that $F_1 \subset G_1$.

Let $z \in F_1$. If $z \in Y$ then $z \in U_1 \subset G_1$. If $z \notin Y$ then there is a unique point $x \in X \setminus A$ such that $z = p(x)$, and since $z \in F_1 \subset K_1$ we get that $x \in J_1 \subset V_1$ and so $x \in V_1 \setminus A$ and hence $z \in G_1$. Similarly, $F_2 \subset G_2$.

Furthermore, we want to show that $G_1 \cap G_2 = \emptyset$. Now, the sets U_1 and U_2 were disjoint by definition. Because $V_1 \cap V_2 = \emptyset$ and $p|(X \setminus A) \to (Z \setminus Y)$ is a homeomorphism, we get that $p(V_1 \setminus A) \cap p(V_2 \setminus A) = \emptyset$. Since $U_1 \subset Y$ and $p(V_2 \setminus A) \subset Z \setminus Y$ we get $U_1 \cap p(V_2 \setminus A) = \emptyset$. Similarly $U_2 \cap p(V_1 \setminus A) = \emptyset$, and hence we get that $G_1 \cap G_2 = \emptyset$.

It remains to show that G_1 and G_2 are open in Z. Since p is an identification map, this is equivalent to showing that $p^{-1}(G_i)$ is open in W for $i = 1, 2$.

On one hand,

$$p^{-1}(G_1) \cap Y = G_1 \cap Y = U_1$$

is open in Y; on the other hand, we get

$$p^{-1}(G_1) \cap X = (V_1 \setminus A) \cup f^{-1}(U_1)$$

since $p^{-1}(U_1) \cap X = f^{-1}(U_1)$. Because $f^{-1}(U_1)$ is open in A there exists an open set H_1 in X such that $f^{-1}(U_1) = A \cap H_1$. Since $p^{-1}(U_1) \cap X \subset J_1 \subset V_1$, we have

$$f^{-1}(U_1) = A \cap H_1 \cap V_1,$$

and so

$$p^{-1}(G_1) \cap X = (V_1 \setminus A) \cup (A \cap H_1 \cap V_1) = (V_1 \setminus A) \cup (H_1 \cap V_1).$$

The sets $V_1 \setminus A$ and $H_1 \cap V_1$ are open in X, and hence $p^{-1}(G_1) \cap X$ is open in X. Hence G_1 is open in Z. Similarly, G_2 is open in Z. Hence Z is normal.

□

Now that we know a little bit about adjunction spaces we can look at what it means to form a new space from an original one by adjoining cells. In this context, an n-cell of the space X will be a subset $e_j^n \subset X$ for which there exists a surjective mapping $f_j : \overline{B}^n \to e_j^n$ such that $f_j|B^n$ is a homeomorphism.

Definition 3.3.8 (Adjoining cells). *Let A be a closed subset of a topological space X. The set X is obtained from A by* adding n-cells, *where $n \geq 0$, if there is a set of cells $\{e_j^n : j \in J\}$ such that*

i) For each $j \in J$, $e_j^n \subset X$

ii) If $\dot{e}_j^n = e_j^n \cap A$ then $(e_j^n \setminus \dot{e}_j^n) \cap (e_{j'}^n \setminus \dot{e}_{j'}^n) = \emptyset$ for $j \neq j'$.

iii) $X = A \cup \bigcup_{j \in J} e_j^n$

iv) X has a topology coherent with $\{A, e_j^n : j \in J\}$, in other words, the topology of X is coinduced by the inclusions of the sets A and e_j^n.

v) For each $j \in J$ there is a map

$$f_j : (\overline{B}^n, S^{n-1}) \to (e_j^n, \dot{e}_j^n)$$

such that $f_j(\overline{B}^n) = e_j^n$, f_j maps $\overline{B}^n \backslash S^{n-1}$ homeomorphically onto $e_j^n \backslash \dot{e}_j^n$ and e_j^n has the topology coinduced by f_j and the inclusion map $\dot{e}_j^n \hookrightarrow e_j^n$.

The map f_j is then called a characteristic map *for e_j^n and $f_j|_{S^{n-1}} : S^{n-1} \to A$ is called an* attaching map *for e_j^n.*

The definition above is most suitable when looking for a CW-complex representation of a given space. Alternatively, one may start in the other end with a given space A and then attach cells to get a space X with certain properties. Because, if we have a space A and an indexed collection of maps $\{g_j : S^{n-1} \to A : j \in J\}$ then we may define a map $g : \dot{\bigcup}_{j \in J} S^{n-1} \to A$ and so there is a space X defined as the adjunction space of the topological sum $A \dot{\cup} \dot{\bigcup}_{j \in J} E_j^n$ where $E_j^n = \overline{B}^n$ for all $j \in J$, by the mapping g. Then the composition of the inclusion map $(E_j^n, S_j^{n-1}) \hookrightarrow (A \dot{\cup} \dot{\bigcup}_{j \in J} E_j^n, A \dot{\cup} \dot{\bigcup}_{j \in J} S_j^{n-1})$ and the projection onto (X, A) is a characteristic map $f_j : (E_j^n, S_j^n) \to (X, A)$ for an n-cell $e_j^n = f_j(E_j^n)$.

Proposition 3.3.9. *If A is T_1/normal and X is obtained from A by adjoining n-cells for some $n \in \mathbb{N}_0$, then X is T_1/normal.*

Proof. Since $\overline{B}^n$ is T_1 and normal, so is the disjoint union $\dot{\bigcup}_{i \in I} \overline{B}_i^n$. Hence it follows from Proposition (3.3.7) that if A is T_1/normal, then so is X. $\square$

Definition 3.3.10 (Strong deformation retract). *Let X be a topological space. The subspace $A \subset X$ is a* strong deformation retract *of X if there is a retraction $r : X \to A$ such that if $i : A \hookrightarrow X$ then $Id_X \simeq ir$ rel A. If $F : Id_X \simeq ir$ rel A then F is a* strong deformation retraction *of X to A.*

Example 3.3.11. *For $n \in \mathbb{N}$, the set $\overline{B}^n \times 0 \cup S^{n-1} \times I$ is a strong deformation retract of $\overline{B}^n \times I$.*

Lemma 3.3.12. *If X is obtained from A by adding n-cells then $X \times 0 \cup A \times I$ is a strong deformation retract of $X \times I$.*

Proof. For each n-cell e_j^n of $X \setminus A$, let $f_j : (\overline{B}^n, S^{n-1}) \to (e_j^n, \dot{e}_j^n)$ be a characteristic map. Let $D : (\overline{B}^n \times I) \times I \to \overline{B}^n \times I$ be a strong deformation retraction of $\overline{B}^n \times I$ to $\overline{B}^n \times 0 \cup S^{n-1} \times I$. Then there is a map

$$D_j : (e_j^n \times I) \times I \to X \times I$$

defined by

$$D_j((f_j(z),t),t') = (f_j \times Id_I)(D(z,t,t')) \text{ when } z \in \overline{B}^n; t, t' \in I.$$

Now there is a map

$$D' : (X \times I) \times I \to X \times I$$

such that $D'|_{(e_j^n \times I)\times I} = D_j$, and $D'(a,t,t') = (a,t)$ for $a \in A; t, t' \in I$, since if $x \in A \cap e_j^n$ then $x \in \dot{e}_j^n$ and so $D_j(x,t,t') = (f_j \times Id_I)(D(z,t,t')) = (f_j \times Id_I)(z,t) = (x,t)$ since $z \in f^{-1}(x) \in S^{n-1}$. Finally D' is a strong deformation retraction of $X \times I$ to $X \times 0 \cup A \times I$ because D is a strong deformation retraction.

(Clearly, $D'_0|_{A\times I} = Id_{A\times I}$, and if $f_j(z) \in e_j^n$, then $D'_0(f_j(z),t) = D_j(f_j(z),t,0) = (f_j \times Id_I)(D(z,t,0)) \overset{\text{D strong def retr}}{=} (f_j \times Id_I)(z,t) = (f_j(z),t)$. On the other hand, D'_1 defines a retraction $r : X \times I \to X \times 0 \cup A \times I$ since $D'_1(f_j(z),t) = D_j(f_j(z),t,1) = (f_j \times Id_I)(D(z,t,1))$ and since D is a strong deformation retraction, $D(z,t,1) \in \overline{B}^n \times 0 \cup S^{n-1} \times I$; thus $(f_j \times Id_I)(D(z,t,1)) \in X \times 0 \cup A \times I$, and so D'_1 defines a retraction r. Hence D' is a strong deformation retraction.) □

Definition 3.3.13 (Cofibration). *A pair of topological spaces (X, A) (that is, A is a subspace of X) has the* homotopy extension property *with respect to a space Y if for all mappings $f : X \to Y$ and homotopies $H : A \times I \to Y$ such that $H(a,0) = f(a)$ for all $a \in A$, there exists a homotopy $F : X \times I \to Y$ such that $F|_{A\times I} = H$ and $F(x,0) = f(x)$ for all $x \in X$.*

The inclusion $i : A \hookrightarrow X$ is a cofibration *if the pair (X, A) has the homotopy extension property with respect to all spaces Y.*

Proposition 3.3.14. *If $X \times 0 \cup A \times I$ is a strong deformation retract of $X \times I$ then $A \hookrightarrow X$ is a cofibration.*

Proof. Let Y be some space, and let $f : X \to Y$ be a mapping and $H : A \times I \to Y$ a homotopy such that $H(a,0) = f(a)$ for all points $a \in A$. These now define a new mapping $g : X \times 0 \cup A \times I \to Y$, which is continuous by Lemma (1.4.1) since both $A \times I$ and $X \times 0$ are closed in $X \times 0 \cup A \times I$. If $r : X \times I \to X \times 0 \cup A \times I$ is a retraction, then $F = g \circ r : X \times I \to Y$ is the wanted homotopy.

□

Corollary 3.3.15. *Since $\overline{B}^n \times 0 \cup S^{n-1} \times I$ is a strong deformation retract of $\overline{B}^n \times I$, it follows that the inclusion $S^{n-1} \hookrightarrow \overline{B}^n$ is a cofibration.*

Similarly, the inclusion $\partial I^n \hookrightarrow I^n$ is a cofibration as well.

Corollary 3.3.16. *If X is obtained from A by adjoining n-cells, then $A \hookrightarrow X$ is a cofibration.*

3.4 CW-complexes

In this section we define CW-complexes and investigate their topological properties.

References: [8], [4]

Definition 3.4.1 (CW-complex). *A* relative CW-complex *consists of a topological space X, a closed subspace A and a sequence of closed subspaces $(X, A)^k$ for $k \in \mathbb{N}_0$ such that*

i) $(X, A)^0$ is obtained from A by adjoining 0-cells.

ii) for $k \geq 1$, $(X, A)^k$ is obtained from $(X, A)^{k-1}$ by adjoining k-cells.

iii) $X = \bigcup_{k \in N_0} (X, A)^k$.

iv) X has a topology coherent with $\{(X, A)^k : k \in \mathbb{N}_0\}$.

The set $(X, A)^k$ is called the k-skeleton of X relative to A. If $X = (X, A)^n$ for some $n \in \mathbb{N}_0$, then we write dimension $(X - A) \leq n$.

An absolute CW-complex *is a CW-complex $(X, \emptyset)$.*

Remark 3.4.2. *Although in the definition we only demand that the topology of a relative CW-complex (X, A) is coherent with the family of k-skeletons $(X, A)^k$ of (X, A), it is actually true that the topology is coherent with the set $\{A, e_j^n : e_j^n$ is a cell in $(X, A)^n\}$:*

Assume that $U \subset X$ is open in (X, A). Then $U \cap A$ is open in A. Assume $n \in \mathbb{N}_0$, then $U \cap (X, A)^n$ is open in $(X, A)^n$ and hence $U \cap e_j^n$ is open in e_j^n for each n-cell e_j^n in $(X, A)^n$.

Conversely, assume that $U \subset X$ such that $U \cap A$ is open in A and $U \cap e_j^n$ is open in each n-cell e_j^n from $(X, A)^n$ for each $n \in \mathbb{N}_0$. Then $U \cap (X, A)^0$ is open in $(X, A)^0$ and by induction $U \cap (X, A)^n$ is open in $(X, A)^n$ for each $n \in \mathbb{N}$, hence U is open in (X, A).

Example 3.4.3. *A simplicial polytope is an absolute CW-complex.*

The following observation is easy to verify:

Remark 3.4.4. *Let $n \in \mathbb{N}_0$. If X is an absolute CW-complex and the topological space Y is obtained by adjoining n-cells via mappings $S^{n-1} \to X$ whose images lie in X^{n-1}, then Y is a CW-complex whose k-skeletons equals X^k for $k < n$ and $X^k \cup$ the new n-cells for $k \geq n$.*

Proposition 3.4.5. *A compact subset of a relative CW-complex (X, A) is contained in the union of A and finitely many cells.*

Proof. Let C be a compact subspace of the relative CW-complex (X, A). Assume that C is not contained in the union of A and any finite colletion of cells of (X, A); then we can find a sequence of points (x_i) in C such that the x_i lie in distinct cells e_i^n. Denote $S = \{x_i : i \in \mathbb{N}\}$

Now the set S is closed in (X, A): $S \cap e_j^0$ is closed in e_j^0 for each 0-cell e_j^0 and so $S \cap (X, A)^0$ is closed in $(X, A)^0$. Furthermore, if $S \cap (X, A)^{n-1}$ is closed in $(X, A)^{n-1}$ for some $n \geq 1$, then $S \cap e_j^n$ contains at most one point which was not contained in $S \cap (X, A)^{n-1}$ for each n-cell e_j^n; hence $S \cap e_j^n$ is closed in e_j^n. It follows that $S \cap (X, A)^n$ is closed in $(X, A)^n$ for all $n \in \mathbb{N}_0$, and hence S is closed in (X, A). In particular, S is closed in the compact set C, and so S is compact.

Similarly, any subset of S is closed, and hence S is discrete. But a discrete infinite set cannot be compact; hence we get a contradiction. Thus C is contained in the union of A and finitely many cells of (X, A). $\square$

Corollary 3.4.6.

- *A compact subset of an absolute CW-complex X is contained in a finite union of cells of X.*
- *A compact subset of a relative CW-complex (X, A) is contained in $(X, A)^k$ for some $k \in \mathbb{N}_0$.*
- *A compact subset of an absolute CW-complex X is contained in X^k for some $k \in \mathbb{N}_0$.*

Lemma 3.4.7. *If $f : X \to Y$ is an identification map and C is a locally compact Hausdorff space, then $f \times Id : X \times C \to Y \times C$ is an identification map.*

Proof. We need to show that if $U \subset Y \times C$ is a subset such that $V = (f \times Id)^{-1}U$ is open, then U is open. Let $(y, c) \in U$ and choose $(x, c) \in V$ such that $f(x) = y$. Now there exists a neighborhood B of c such that

$\overline{B}$ is compact and $\{x\} \times \overline{B} \subset V$. Let $W = \{z \in X : \{z\} \times \overline{B} \subset V\} = \{z \in X : \{f(z)\} \times \overline{B} \subset U\}$. Now V is open and $\overline{B}$ and each $\{z\}$ are compact, so for each z there exist open sets U_z of Y and V_z' of C such that $\{z\} \times \overline{B} \subset U_z \times V_z' \subset V$. Hence $W = \cup\{U_z : z \in W\}$ is open, and $f^{-1}(f(W)) = W$ - thus $f(W)$ is open, since f is an identification map. Hence U contains the open neighborhood $B \times f(W)$ of (y, c); thus U is open. $\square$

Proposition 3.4.8. *If (X, A) is a relative CW-complex then a function $H : (X, A) \times I \to Y$ is continuous if and only if it is continuous on $A \times I$ and on $e_j^n \times I$ for each cell e_j^n of (X, A).*

Proof. The topology of the CW-complex (X, A) is coherent with $\{A, e_j^n\}$, as we already know. That is, we may define an identification map $p : \dot{\bigcup}_{j\in J} e_j^n \dot{\cup} A \to (X, A)$ from the disjoint union of A and the cells of (X, A) onto (X, A). By Lemma (3.4.7) the mapping $p \times I : \dot{\bigcup}_{j\in J} e_j^n \times I \dot{\cup} A \times I \to (X, A) \times I$ is also a quotient map. Hence any mapping $H : (X, A) \times I \to Y$ for some space Y is continuous if and only if the corresponding map $H' : \dot{\bigcup}_{j\in J} e_j^n \times I \dot{\cup} A \times I \to Y$ is continuous - that is, if H is continuous on $A \times I$ and on $e_j^n \times I$ for each cell e_j^n of (X, A). $\square$

Proposition 3.4.9. *If (X, A) is a relative CW-complex then the inclusion map $A \hookrightarrow X$ is a cofibration.*

Proof. Let $f : (X \times 0) \cup (A \times I) \to Y$ be a mapping - we now need to show that f can be extended to $X \times I$. By Corollary (3.3.16) there exists an extension $f_0 : X \times 0 \cup (X, A)^0 \times I \to Y$. Similarly, if we have managed to extend f to a mapping $f_{n-1} : X \times 0 \cup (X, A)^{n-1} \times I \to Y$ then, again by Corollary (3.3.16) there exists an extension $f_n : X \times 0 \cup (X, A)^n \times I \to Y$.

Now define $F : X \times I \to Y$ by setting $F(x, t) = f_n(x, t)$ where $x \in (X, A)^n$. Now since F is continuous on $(X, A)^n \times I$ for all $n \in \mathbb{N}_0$, it is continuous on $A \times I$ and on $e_j^n \times I$ for every cell e_j^n of (X, A); hence by Proposition (3.4.8) it is continuous on $(X, A) \times I$. $\square$

Lemma 3.4.10. *For a mapping $\alpha : (\overline{B}^n, S^{n-1}, s_0) \to (X, A, x_0)$ we have that $[\alpha] = 0$ in $\pi_n(X, A, x_0)$ if and only if α is homotopic relative to S^{n-1} to some map $\overline{B}^n \to A$.*

Proof. "$\Rightarrow$" Assume that $[\alpha] = 0$ in $\pi_n(X, A, x_0)$. Then there is a homotopy

$$H : (\overline{B}^n, S^{n-1}, s_0) \times I \to (X, A, x_0)$$

from α to $\epsilon_{x_0} : \overline{B}^n \to \{x_0\} \hookrightarrow X$.

We define a function $H' : (\overline{B}^n, S^{n-1}, s_0) \times I \to (X, A, x_0)$ by setting

$$H'(z,t) = \begin{cases} H(\frac{z}{1-\frac{t}{2}}, t), & 0 \le \|z\| \le 1 - \frac{t}{2} \\ H(\frac{z}{\|z\|}, 2 - 2\|z\|) & 1 - \frac{t}{2} \le \|z\| \le 1 \end{cases}$$

Clearly H' is continuous, since when $\|z\| = 1 - \frac{t}{2}$, then $H(\frac{z}{\|z\|}, 2-2\|z\|) = H(\frac{z}{1-\frac{t}{2}}, 2 - 2(1-\frac{t}{2})) = H(\frac{z}{1-\frac{t}{2}}, t)$. Note also that

$$\begin{aligned}
&H'(z,0) = H(z,0) = \alpha(z) \ \forall \ z \in \overline{B}^n, \\
&H'(z,1) = \begin{cases} H(2z,1) = x_0 \in A, & \|z\| \le \frac{1}{2} \\ H(\frac{z}{\|z\|}, 2 - 2\|z\|) \in A, & \|z\| \ge \frac{1}{2} \text{ because } \frac{z}{\|z\|} \in S^{n-1}, \end{cases} \\
&H'(z,0) = \alpha(z) \in A \text{ if } z \in S^{n-1}, \\
&H'(z,t) = H(\tfrac{z}{\|z\|}, 2 - 2\|z\|) = H(z,0) = \alpha(z) \text{ if } z \in S^{n-1} \text{ and } t \ne 0 \text{ because } \|z\| = 1.
\end{aligned}$$

Thus we see that H' is a homotopy rel S^{n-1} from α to some map $\overline{B}^n \to A$, just like we wanted.

"$\Leftarrow$" Now we assume that there is mapping $\alpha' : (\overline{B}^n, S^{n-1}, s_0) \to (X, A, x_0)$ such that $\alpha \simeq \alpha'$ rel S^{n-1} and $\alpha'(\overline{B}^n) \subset A$. Then $[\alpha] = [\alpha']$ in $\pi_n(X, A, x_0)$. A homotopy $H : (\overline{B}^n, S^{n-1}, s_0) \times I \to (X, A, x_0)$ from α' to the constant map ϵ_{x_0} is defined by

$$H(z,t) = \alpha'\left((1-t)z + ts_0\right),$$

and so $[\alpha] = [\alpha'] = 0$.

$\square$

Definition 3.4.11 (n-connectedness). *Let $n \in \mathbb{N}_0$. A topological space X is* n-connected *if every mapping*

$$f : S^k \to X,$$

where $k \le n$, can be extended to a mapping $\overline{B}^{k+1} \to X$.

A pair (X, A) is n-connected if every mapping

$$f : (\overline{B}^k, S^{k-1}) \to (X, A)$$

is homotopic rel S^{k-1} to some mapping $\overline{B}^k \to A$ for any k such that $0 \le k \le n$.

In the case $n = 0$ the pair $(\overline{B}^0, S^{-1})$ consists of a single point and the empty set, and so 0-connectedness means that any point in X can be connected by a path to some point in A.

Note that 0-connectedness in the case of a single space $X = (X, \emptyset)$ is equivalent to path-connctedness, and in the case of a pair (X, A) of spaces means that every path component of X intersects with A.

Lemma 3.4.12. *A topological space X is n-connected for some $n \in \mathbb{N}$ if and only if it is path connected and $\pi_k(X, x_0) = 0$ for all $x_0 \in X$ and for all $k = 1, ..., n$.*

Proof. "$\Rightarrow$" Assume that the topological space X is n-connected for some $n \in \mathbb{N}$. Then any mapping $f : S^0 = \{-1, 1\} \to X$ can be extended to a mapping $\overline{B}^1 = [-1, 1] \to X$ - in other words, X is path connected. In addition to that, any map $f : (S^k, p_0) \to (X, x_0)$ can be extended to a map $g : \overline{B}^{k+1} \to X$ when $1 \leq k \leq n$. Now define a homotopy

$$h : S^k \times I \to X$$

by setting

$$h(x, t) = g((1-t)x + tp_0)$$

where p_0 is some point in S^n. Then h is a homotopy from f to c_{x_0}, and $h(p_0, t) = g(p_0)$ for all $t \in I$ and hence h is a homotopy rel p_0.

It follows that $[f] = [c_{x_0}] = 0$ for all maps $f : (S^k, p_0) \to (X, x_0)$ and hence $\pi_k(X, x_0) = 0$ for all $k = 1, ..., n$.

"$\Leftarrow$" Assume that the topological space X is path connected and that $\pi_k(X, x_0) = 0$ for all $k = 1, ..., n$.

Since X is path-connected any map $f : S^0 = \{-1, 1\} \to X$ can be extended to a mapping $[-1, 1] = \overline{B}^1 \to X$.

Now assume $0 < k \leq n$, and let $f : S^k \to X$. Since $\pi_k(X, x_0) = 0$ for all $x_0 \in X$, there is a homotopy $h : S^k \times I \to X$ such that $h(x, 0) = f(x)$ for all $x \in S^n$ and $h(x, 1) = c_{x_0}$ where c_{x_0} is a constant map. Now, since $S^k \hookrightarrow \overline{B}^{k+1}$ is a cofibration there exists an extension F of the mapping $H : S^k \times I \cup \overline{B}^{k+1} \times 1$ given by $H(x, 1) = x_0$ when $x \in \overline{B}^{k+1}$, $H(x, t) = h(x, t)$ when $(x, t) \in S^k \times I$. Now F_0 is an extension of f over $\overline{B}^{k+1}$, and hence X is n-connected.

□

Lemma 3.4.13. *A pair of spaces (X, A) is n-connected if and only if the following holds: Every path component of X intersects A and for every point $a \in A$ and every $1 \leq k \leq n$,*

$$\pi_k(X, A, a) = 0.$$

Proof. "$\Rightarrow$" Assume that the topological pair (X, A) is n-connected, and let $x \in X$. Now the pair $(\overline{B}^0, S^{-1})$ consist in fact of a singleton set $\{1\}$ and the empty set, which we may treat as the singleton set only.

Now the mapping $f : \{1\} \to X$ defined by $1 \mapsto x$ for some $x \in X$ is homotopic to some mapping $g : \{1\} \to X$ such that $f(1) = a \in A$; hence there is a path in X from x to a and hence the path component of x in X intersects A.

Furthermore, let $0 \leq k \leq n$. Then by the definition, any map

$$f : (\overline{B}^k, S^{k-1}, s_0) \to (X, A, a)$$

is homotopic rel S^{k-1} to some mapping $g : (\overline{B}^k, S^{k-1}) \to (X, A)$ such that $g(\overline{B}^k) \subset A$, and thus by Lemma (3.4.10), $[f] = 0$ in $\pi_k(X, A, a)$. Hence $\pi_k(X, A, a) = 0$.

"$\Leftarrow$" Now assume that every path component intersects with A and that $\pi_k(X, A, x_0) = 0$ for $1 \leq k \leq n$. Then, given a map $f : (\overline{B}^k, S^{k-1}, s_0) \to (X, A, x_0)$ where $1 \leq k \leq n$, this map is homotopic rel S^{k-1} to a map $\overline{B}^k \to A$ by Lemma (3.4.10). Furthermore, if $f : (\overline{B}^0, S^{-1}) = (\{1\}, \emptyset) \to (X, A)$; by the assumption there is a path $\gamma : I \to X$ from $f(1)$ to some point $a \in A$; hence the map $H : \{1\} \times I \to X$ defined by $H(1, t) = \gamma(t)$ defines a homotopy from f to a map $\overline{B}^0 \to A$. Hence the pair (X, A) is n-connected.

□

Example 3.4.14. *For all $n \in \mathbb{N}$ the pair $(\overline{B}^n, S^{n-1})$ is $n-1$-connected.*

Proof. See [8]. □

Lemma 3.4.15. *Let X be obtained from A by adding n-cells and let (Y, B) be a pair of spaces such that*

- *$\pi_n(Y, B, b) = 0$ for all $b \in B$ if $n \geq 1$;*
- *every point of Y can be joined to B by a path if $n = 0$.*

Then any map $(X, A) \to (Y, B)$ is homotopic rel A to some map $X \to B$.

Proof. $\mathbf{n = 0}$
Let $f : (X, A) \to (Y, B)$ be a map, and let the 0-cells be points e_j^0, where $j \in J$. Now for each $y_j = f(e_j^0)$ there is a point $b_j \in B$ such that y_j and b_j may be connected with a path $\alpha_j : I \to Y$. Now define $H : X \times I \to Y$ by setting

$$\begin{aligned} H(a,t) &= f(a) \quad && \forall\, t \in I, a \in A \\ H(e_j^0, t) &= \alpha_j(t) \quad && \forall\, t \in I, j \in J \end{aligned}$$

Since X is the topological sum of A and the discrete space $\{e_j^0 : j \in J\}$, H is continuous, and $H_1(X) \subset B$. Hence H is the wanted homotopy.

$\mathbf{n > 1}$
Let the characteristic map of each n-cell e_j^n be $f_j : (\overline{B}_j^n, S^{n-1}) \to (e_j^n, \dot{e}_j^n)$, and let $f : (X, A) \to (Y, B)$ be any map. For each n-cell e_j^n there is a well-defined map

$$f \circ f_j : (\overline{B}^n, S^{n-1}, e_{n+1}) \to (Y, B, f(f_j(e_{n+1})))$$

and since $\pi_n(Y, B, f(f_j(e_{n+1}))) = 0$ we have $[f \circ f_j] = 0$ and so by Lemma (3.4.10) $f \circ f_j$ is homotopic rel S^{n-1} to some map $\overline{B}^n \to B$. Denote this homotopy H^j, where $H_0^j = f \circ f_j$ and $H_1^j(\overline{B}^n) \subset B$.

Define $H : X \times I \to Y$ by setting

$$\begin{aligned} H(a,t) &= a \quad && \forall\, a \in A, t \in I \\ H(f_j(x), t) &= H^j(x,t) \quad && \forall\, f_j(x) \in e_j^n \text{ where } x \in \overline{B}^n. \end{aligned}$$

The function H is well-defined since H^j was a homotopy rel S^{n-1} for all $j \in J$, and H is continuous on $A \times I$ and on $e_j^n \times I$ for all $j \in J$. It follows from Proposition (3.4.8) that H is continuous, and hence H is the wanted homotopy.

□

Lemma 3.4.16. *Let (X, A) be a relative CW-complex with dimension$(X - A) \leq n$ and let (Y, B) be n-connected. Then any map from (X, A) to (Y, B) is homotopic rel A to some map from X to B.*

Proof. We will prove this lemma by induction. First assume that $n = 0$. Since X is obtained from A by adding 0-cells, and since the 0-connectedness of (Y, B) means that every point of Y may be joined by a path to some point of B, we may apply Lemma (3.4.15) to get that any map $f : (X, A) \to (Y, B)$ is homotopic rel A to some map $X \to B$.

Now assume that the claim holds for all $n < m$, and assume that $dimension(X - A) \leq m$ and that (Y, B) is m-connected. Let

$$f : (X, A) \to (Y, B)$$

be any mapping. Now, by the induction assumption, the restriction

$$f| : \big((X, A)^{m-1}, A\big) \to (Y, B)$$

is homotopic rel A to some mapping $(X, A)^{m-1} \to B$, and by Proposition (3.4.9) this homotopy has an extension $H : X \times I \to Y$ such that $H_0 = f$, while $H_1 : (X, (X, A)^{m-1}) \to (Y, B)$. But now by Lemma (3.4.15), since X is obtained from $(X, A)^{m-1}$ by adding m-cells, the mapping H_1 is homotopic to some map g where $g(X) \subset B$ - that is, $f \simeq H_1 \simeq g$ and g maps X into B. $\square$

Corollary 3.4.17. *Let (X, A) be a relative CW-complex and let (Y, B) be n-connected for all $n \in \mathbb{N}$. Then any map $(X, A) \to (Y, B)$ is homotopic rel A to a map from X to B.*

Proof. Let $f : (X, A) \to (Y, B)$. By Lemma (3.4.16) there exists a homotopy $h^0 : (X, A)^0 \times I \to Y$ relative to A such that $h^0_0 = f|_{(X,A)^0}$ and $h^0_1((X, A)^0) \subset B$ since $dimension((X, A)^0 - A) = 0$. But because $(X, A)^0 \hookrightarrow X$ is a cofibration by Proposition (3.4.9) there exists an extension $H_0 : (X, A) \times I \to (Y, B)$ of h^0 such that $H_0(x, 0) = f(x)$ for all $x \in X$. Now H_0 is a homotopy relative to A.

Now assume that there exist homotopies $H_k : (X, A) \times I \to (Y, B)$ for $k < n$ such that

a) $H_{k-1}(x, 1) = H_k(x, 0)$ for $x \in X$.

b) H_k is a homotopy rel $(X, A)^{k-1}$.

c) $H_k((X, A)^k \times 1) \subset B$.

Then considering the map $g : ((X, A)^n, (X, A)^{n-1}) \to (Y, B)$ where $g(x) = H_{n-1}(x, 1)$ for all $x \in (X, A)^n$ there exists, according to Lemma (3.4.16), a homotopy $h^n : (X, A)^n \times I \to Y$ relative to $(X, A)^{n-1}$ such that $h^n_0 = g$ and $h^n_1((X, A)^n) \subset B$. But by Proposition (3.4.9) $(X, A)^n \hookrightarrow X$ is a cofibration and hence there exists an extension $H_n : (X, A) \times I \to (Y, B)$ of h^n such that $H_n(x, 0) = H_{n-1}(x, 1)$. Now H_n is a homotopy relative to $(X, A)^{n-1}$.

Hence we get a sequence of homotopies $H_k : (X, A) \times I \to (Y, B)$ such that

i) $H_0(x, 0) = f(x)$ for $x \in X$.

ii) $H_k(x, 1) = H_{k+1}(x, 0)$ for $x \in X$.

iii) H_k is a homotopy rel $(X,A)^{k-1}$.

iv) $H_k((X,A)^k \times 1) \subset B$.

Now we may define a homotopy $H : (X,A) \times I \to (Y,B)$ by setting

$$H(x,t) = H_{k-1}\left(x, \frac{t-(1-\frac{1}{k})}{(\frac{1}{k}-\frac{1}{k+1})}\right) \quad 1-\frac{1}{k} \le t \le 1-\frac{1}{k+1}$$

$$H(x,1) = H_k(x,1) \quad x \in (X,A)^k$$

Now using Lemma (1.4.4) one sees that H is continuous on $(X,A)^n \times I$ for all $n \in \mathbb{N}_0$ and hence it is continuous on $(X,A) \times I$. Furthermore, $H(x,0) = H_0(x,0) = f(x)$ for all $x \in X$ while $H(x,1) \subset B$ for all $x \in X$. □

Lemma 3.4.18. *If the space X is obtained from A by adjoining n-cells, then the pair (X,A) is $(n-1)$-connected.*

Proof. First consider a simpler case: Assume that X is obtained from A by adjoining the n-cell e^n to A. We show that the pair (X,A) is n-connected.

Let $x_0 \in \mathrm{Int} e^n$. Define a subset Y of X by setting

$$Y = A \cup e^n \setminus \{x_0\} = X \setminus \{x_0\},$$

and let $f : (\overline{B}^k, S^{k-1}) \to (X,A)$ be a mapping where $0 \le k < n$. We may identify (up to homeomorphism) $\overline{B}^k$ with the standard k-simplex Δ_k and S^{k-1} with its boundary. Then Y and $\mathrm{Int} e^n$ intersect $f(\Delta_k)$ in open subsets of $f(\Delta_k)$, hence $f^{-1}(Y)$ and $f^{-1}(\mathrm{Int} e^n)$ are open subsets of Δ_k. By Lebesgue's covering theorem we may subdivide Δ_k into finitely many smaller simplices such that each simplex belongs either to $f^{-1}(Y)$ or to $f^{-1}(\mathrm{Int} e^n)$. This subdivision corresponds to a finite simplicial complex which we may call K whose underlying set is the same as that of Δ_k, and since the complex is finite the topology on the corresponding polytope $|K|$ is the same as that on Δ_k - hence $|K| = \Delta_k$ as topological spaces.

Now for each simplex $s \in K$ either $f(|s|) \subset Y$ or $f(|s|) \subset \mathrm{Int} e^n$.

Let A' be the subpolytope of $|K|$ containing those simplices $s \in K$ such that $f(|s|) \subset Y$, and let B be the subpolytope containing such simplices $s \in K$ that $f(|s|) \subset \mathrm{Int} e^n$. Then $S^{k-1} \subset A'$ and $\Delta_k = A' \cup B$.

Denote $B' = B \cap A'$ and note that (B, B') is a relative CW-complex with $dimension(B - B') \leq k \leq n - 1$. We consider the restriction

$$f|_{(B,B')} : (B, B') \to (\mathrm{Int}e^n, \mathrm{Int}e^n \setminus \{x_0\}).$$

The pair $(\mathrm{Int}e^n, \mathrm{Int}e^n \setminus \{x_0\})$ is homeomorphic to $(B^n, B^n \setminus \{0\})$, and thus their homotopy groups are the same as that of $(\overline{B}^n, S^{n-1})$. By Example (3.4.14) $(\overline{B}^n, S^{n-1})$ is $n-1$-connected, and thus by Lemma (3.4.16) we have that $f|_{(B,B')}$ is homotopic rel B' to a map $B \to \mathrm{Int}e^n \setminus \{x_0\}$.

This can be extended to a homotopy rel A' from f to a mapping $f' : \overline{B}^k \subset Y$.

Now A is a strong deformation retract of Y (in a similar way that S^{n-1} is a strong deformation retract of $\overline{B}^n \setminus \{0\}$). It follows that f' is homotopic rel S^{k-1} to a mapping f'' such that $f''(\overline{B}^k) \subset A$. But then $f \simeq f''$ rel S^{k-1}, and it follows that (X, A) is $n-1$-connected.

Now we return to the original question. Let X be obtained from A by adjoining n-cells (an arbitrary amount of such cells), and let $f : (\overline{B}^k, S^{k-1}) \to (X, A)$ where $0 \leq k < n$. Now since $\overline{B}^k$ is compact, its image $f(\overline{B}^k)$ is compact and so by Lemma (3.4.5) it is contained in the union of A and finitely many n-cells - that is, $f(\overline{B}^k) \subset A \cup e_1^n \cup \ldots \cup e_m^n$. Hence we may write

$$f : (\overline{B}^k, S^{k-1}) \to (X^{(m)}, A)$$

where $X^{(m)}$ is the space obtained by adjoining only the cells $e_1^n, \ldots, e_m^n$. But although we adjoin all cells at once we would obtain exactly the same space by adjoining first e_1^n, then e_2^n and so on. Denote by $X^{(i)}$ the space obtained by adjoining cells e_1^n,...,e_i^n. Now by our previous calculations the pair $(X^{(m)}, X^{(m-1)})$ is $(n-1)$-connected, and so there exists a mapping $f^m : \overline{B}^k \to X^{(m-1)}$ such that $f \simeq f^m$ rel S^{k-1}. Similarly we construct $f^i : \overline{B}^k \to X^{(i-1)}$ for all $i = 1, \ldots, m-1$ ($X^{(0)} = A$) and thus we obtain $f \simeq f^m \simeq f^{m-1} \simeq \ldots \simeq f^1$ where all homotopies are rel S^{k-1} and $f^1(\overline{B}^k) \subset A$. It follows from this that (X, A) is $n-1$-connected.

□

Theorem 3.4.19. *For any relative CW-complex* (X, A) *the pair* $(X, (X, A)^n)$ *is n-connected for all* $n \in \mathbb{N}_0$.

Proof. Let (X, A) be a relative CW-complex, and let $n \in \mathbb{N}_0$. We will start the proof of the theorem by proving that for any $m > n$, the relative CW-complex $((X,A)^m,(X,A)^n)$ is n-connected by induction on m.

If $m=n+1$, then $((X,A)^m,(X,A)^n)$ is n-connected by Lemma (3.4.18).

Now make the inductive assumption that $((X,A)^{m-1},(X,A)^n)$ is n-connected, and let $f:(\overline{B}^k,S^{k-1})\to((X,A)^m,(X,A)^n)$, where $0\leq k\leq n$. Then we may write

$$f:(\overline{B}^k,S^{k-1})\to\big((X,A)^m,(X,A)^{m-1}\big)$$

and $((X,A)^m,(X,A)^{m-1})$ is $m-1$-connected by Lemma (3.4.18) and thus it is also n-connected. It follows that there exists a mapping $g:\overline{B}^k\to(X,A)^{m-1}$ such that $f\simeq g$ rel S^{k-1}. But then $g:(\overline{B}^k,S^{k-1})\to((X,A)^{m-1},(X,A)^n)$ and by the inductive assumption $((X,A)^{m-1},(X,A)^n)$ is n-connected so there exists a mapping $h:\overline{B}^k\to(X,A)^n$ such that $g\simeq h$ rel S^{k-1}.

It follows that

$$f\simeq h \text{ rel } S^{k-1}$$

and it follows that $((X,A)^m,(X,A)^n)$ is n-connected.

Now let's return to the original task. Assume that $f':(\overline{B}^k,S^{k-1})\to((X,A),(X,A)^n)$ where $0\leq k\leq n$. Now since $\overline{B}^k$ is compact, so is $f(\overline{B}^k)$ and thus by Corollary (3.4.6) $f(\overline{B}^k)\subset(X,A)^m$ for some $m\in\mathbb{N}_0$. We may assume that $m>n$.

Then we may consider the function f' as a function

$$f':(\overline{B}^k,S^{k-1})\to((X,A)^m,(X,A)^n)$$

and so by the previous comment there exists a mapping $g':\overline{B}^k\to(X,A)^n$ such that $f'\simeq g'$ rel S^{k-1}. Hence $(X,(X,A)^n)$ is n-connected.

□

3.5 Weak homotopy equivalence

The notion of weak homotopy equivalence is generally weaker than that of homotopy equivalence, and may be easier to prove. In this section we will show that in the case of a CW complex the two are actually equivalent.

Reference: [8]

Definition 3.5.1 (n-equivalence, weak homotopy equivalence). *Let X and Y be topological spaces and let $n \in \mathbb{N}$. A mapping $f : X \to Y$ is an* n-equivalence *if f induces a $1-1$ correspondence between the path components of X and Y and if for each $x \in X$ the induced map*

$$f_* : \pi_q(X, x) \to \pi_q(Y, f(x))$$

is an isomorphism when $0 < q < n$ and an epimorphism when $q = n$.

A mapping $f : X \to Y$ is a weak homotopy equivalence *or an* ∞-equivalence *if it is an n-equivalence for all $n \geq 1$.*

A weak homotopy equivalence is not generally a homotopy equivalence:

Example 3.5.2. *Let $A_1 = \mathbb{N}_0$ and $A_2 = \{0, \frac{1}{n} : n \in \mathbb{N}\}$ with their subspace topologies from $\mathbb{R}$. Now A_2 is not a CW complex because it does not have the discrete topology.*

Let $f : A_1 \to A_2$ be defined by $f(0) = 0, f(n) = \frac{1}{n}$. Now f is continuous since A_1 is discrete, and $f_* : \pi_0(A_1) \to \pi_0(A_2)$ is clearly a bijection. Furthermore, if $f : S^k \to A_i$, $i = 1, 2$ is continuous then f is the constant map since its image must be connected. Hence $\pi_k(A_1) = 0 = \pi_k(A_2)$ for all $k \in \mathbb{N}$. It follows that f_* is an isomorphism for all $k \in \mathbb{N}$ and hence f is a weak homotopy equivalence. However, f is not a homotopy equivalence:

Assume that g is a homotopy inverse of f; then if $a \in A_1$ and $H : A_1 \times I \to A_1$ such that $H : (g \circ f) \simeq id_{A_1}$; then $H_t(a) : I \to A_1$ defines a path in A_1 from $(g \circ f)(a)$ to a and since A_1 is totally disconnected it follows that $(g \circ f)(a) = a$. Hence $g \circ f = Id_{A_1}$. Similarly $f \circ g = Id_{A_2}$, and so f is a homeomorphism. But A_1 and A_2 are not homeomorphic since A_2 is compact and A_1 is not.

Thus f is a weak homotopy equivalence, but not a homotopy equivalence.
□

In order to prove the following lemma we need the notion of a mapping cylinder Z_f of a function $f : X \to Y$. We define Z_f as the quotient space of the topological sum of $X \times I$ and Y by identifying every point $(x, 1) \in X \times I$ with the corresponding point $f(x) \in Y$.

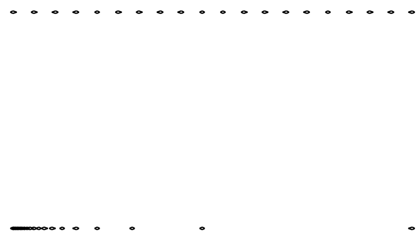

Figure 3.1: Top: A piece of the space A_1. Bottom: The space A_2.

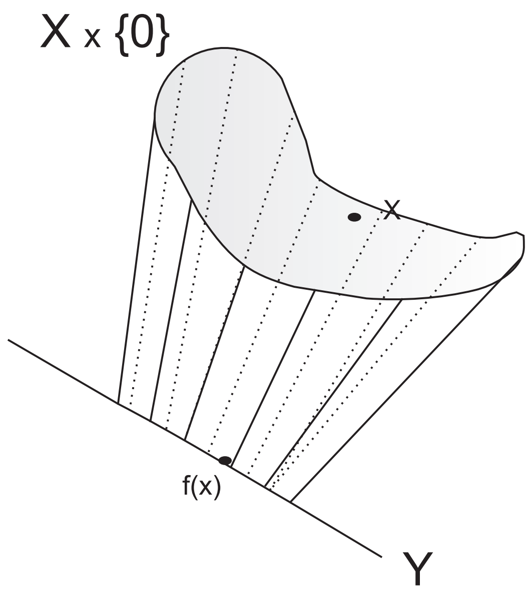

Figure 3.2: The mapping cylinder Z_f of a mapping $f : X \to Y$.

In other words Z_f consists of equivalence classes $[x,t] = \{(x,t)\}$ if $t \in [0,1[$, $[x,1] = [f(x)] = [y] = \{y,(x,1) : f(x) = y\}$ if $y \in f(X)$ and $[y]$ if $y \in Y \setminus f(X)$. Then there is an imbedding $i : X \to Z_f$ defined by $x \mapsto [x,0]$ and an imbedding $j : Y \to Z_f$ defined by $y \mapsto [y]$. By means of these imbeddings X and Y may be viewed as subspaces of Z_f.

Furthermore, we may define a retraction $r : Z_f \to Y$ by setting $r([x,t]) = [f(x)]$ for all $x \in X$ and $r([y]) = [y]$ for all $y \in Y$.

Now it is clear that $r \circ j : Y \hookrightarrow Z_f \to Y = Id_Y$, and furthermore the function $H : Z_f \times I \to Z_f$ defined by

$$\begin{array}{ll} H([x,t],s) = [x, t + s(1-t)], & \forall\, x \in X \text{ and } \forall\, t,s \in I \\ H([y],s) = [y], & \forall\, y \in Y \text{ and } \forall\, s \in I. \end{array}$$

defines a homotopy $H : Id_{Z_f} \simeq j \circ r$, since $H_0([x,t]) = [x,t]$ and $H_1([x,t]) = [x,1] = [f(x)]$. Hence r is a homotopy equivalence.

Naturally, the properties of the mapping cylinder Z_f depend on the properties of the mapping f, and in particular:

Proposition 3.5.3. *If the mapping f is an n-equivalence, then the pair (Z_f, X) is n-connected.*

Proof. Let $f : X \to Y$ be a mapping, and let Z_f be the mapping cylinder of f. Then $f = r \circ i$ where i and r were defined above, and where r is a homotopy equivalence. Hence f is an n-equivalence if and only if $i : X \hookrightarrow Z_f$ is an n-equivalence. Now, if f is an n-equivalence then i is as well, so consider the exact homotopy sequence of the pair (Z_f, X):

$$\ldots \to \pi_n(X, x_0) \xrightarrow{i_*} \pi_n(Z_f, x_0) \xrightarrow{j_*} \pi_n(Z_f, X, x_0) \xrightarrow{\delta} \pi_{n-1}(X, x_0) \xrightarrow{i_*} \pi_{n-1}(Z_f, x_0) \to \ldots$$

Now $Im(\delta) = Ker(i_*) = 0$, and since i_* is surjective, $Ker(\delta) = Im(j_*) = Im(j_* \circ i_*) = 0$ by the exactness of the sequence; hence δ is injective and so it follows that $\pi_n(Z_f, X, x_0) = 0$. Similarly $\pi_k(Z_f, X, x_0) = 0$ whenever $1 \leq k \leq n$.

Furthermore, since f induces a 1-1 correspondence between the path components of X and Y then every point of Z_f can be joined to some point of X by a path. It follows that (Z_f, X) is n-connected.

□

Lemma 3.5.4. *Let $f : X \to Y$ be an n-equivalence (n finite or infinite) and let (P,Q) be a relative CW-complex with dimension$(P - Q) \leq n$. Given maps*

$$g : Q \to X, \qquad h : P \to Y$$

such that $h|_Q = f \circ g$, *there exists a map*

$$g' : P \to X$$

such that $g'|Q = g$ *and* $f \circ g' \simeq h$ *rel* Q.

Proof. Denote by Z_f the mapping cylinder of f with inclusion maps $i : X \hookrightarrow Z_f$ and $j : Y \hookrightarrow Z_f$, and with the retraction $r : Z_f \to Y$ which is a homotopy inverse of j.

Now recall the homotopy $H : Z_f \times I \to Z_f$ from above and use it to form a new homotopy $H' : Q \times I \to Z_f$ by setting

$$H' = H \circ (i \times Id_I) \circ (g \times Id_I).$$

Then $H'(q, 0) = (H \circ (i \times Id_I))(g(q), 0) = H([g(q), 0], 0) = [g(q), 0] = i(g(q)) = (i \circ g)(q)$ and $H'(q, 1) = (H \circ (i \times Id_I))(g(q), 1) = H([g(q), 0], 1) = [g(q), 1] = j(f(g(q))) = (j \circ f \circ g)(q)$, hence $H' : i \circ g \simeq j \circ f \circ g = j \circ h|_Q$. Furthermore, the composition $r \circ H'$ is invariant with respect to $t \in I$. That is, $r \circ H'$ is a homotopy rel Q.

By Lemma (3.4.9) there exists a map $h' : P \to Z_f$ such that $h'|_Q = i \circ g$ and such that $r \circ h' \simeq r \circ j \circ h$ rel Q. We may then consider h' as a mapping $(P, Q) \to (Z_f, X)$. Because (Z_f, X) is n-connected and $dimension(P - Q) \leq n$ we get from Lemma (3.4.16) (or from Corollary (3.4.17) in the infinite-dimensional case) that h' is homotopic rel Q to some mapping $g' : P \to X$. Then $g'|_Q = g$ and

$$f \circ g' = r \circ i \circ g' \simeq r \circ h' \simeq r \circ j \circ h = h$$

where all the homotopies are rel Q. Hence g' is exactly as we wanted it to be.

□

In the set of all functions $f : X \to Y$ we may define an equivalence relation by setting $f \sim g \Leftrightarrow f \simeq g$. The equivalence class of f may then be denoted $[f]$. The set of all such equivalence classes $[f]$ is from now on denoted $[X; Y]$ - that is, $[X; Y] = \{[f] : f : X \to Y\}$.

Lemma 3.5.5. *Let* $f : X \to Y$ *be a weak homotopy equivalence, and let* P *be a CW-complex. Then the induced map*

$$f_* : [P; X] \to [P; Y]$$

defined by $[g] \mapsto [f \circ g]$ *is a bijection.*

Proof. We apply Lemma (3.5.4) to the relative CW-complex $(P, \emptyset)$. Assume that $h : P \to Y$. Now there is a mapping $g' : P \to X$ such that $f \circ g' \simeq h$, or in other words, there is a class $[g'] \in [P; X]$ such that $f_*([g']) = [f \circ g'] = [h] \in [P; Y]$. Hence f_* is surjective.

To prove injectivity, assume that we have two mappings $g_0, g_1 : P \to X$ such that $f \circ g_0 \simeq f \circ g_1$ - then there is a map $g : P \times \dot{I} \to X$ such that $g(x, 0) = g_0(x)$ and $g(x, 1) = g_1(x)$ for $x \in P$ and a map $H : P \times I \to Y$ such that $H|P \times \dot{I} = f \circ g$. Now by applying Lemma (3.5.4) to the relative CW-complex $(P \times I, P \times \dot{I})$ we find that there is a mapping $g' : P \times I \to X$ such that $g'|P \times \dot{I} = g$. Then $g' : g_0 \simeq g_1$ and so $[g_0] = [g_1]$. Thus f_* is injective.

□

Proposition 3.5.6. *A mapping between CW-complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Proof. "$\Leftarrow$" Let X and Y be topological spaces, and let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. Then $(g \circ f)(x)$ belongs to the same path component as x for all $x \in X$; hence $(g \circ f)_* : \pi_0(X, x_0) \to \pi_0(X, x_0)$ is the identity mapping. Similarly $(f \circ g)_* = Id_{\pi_0(Y, y_0)}$, and so f induces a $1-1$-correspondence between the path components of X and Y.

Furthermore, by Proposition (3.1.3), the induced map

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is a group isomorphism for all $n \geq 1$ and all base points $x_0 \in X$. Hence f is a weak homotopy equivalence.

"$\Rightarrow$" Now assume that $f : X \to Y$ is a weak homotopy equivalence. Then by Lemma (3.5.5) f induces bijections

$$f_* : [Y; X] \to [Y; Y], \qquad f_* : [X; X] \to [X; Y].$$

Now, if $g : Y \to X$ is a mapping such that $f_*([g]) = [Id_Y]$, then since $f_*([g]) = [f \circ g] = [Id_Y]$ it follows that $f \circ g \simeq Id_Y$. Furthermore,

$$f_*([g \circ f]) = [f \circ (g \circ f)] = [(f \circ g) \circ f] = [Id_Y \circ f] = [f \circ Id_X] = f_*([Id_X]),$$

and hence by the injectivity of f_* it must be true that $g \circ f \simeq Id_X$.
It follows that f is a homotopy equivalence.

□

3.6 A metrizable ANR is homotopy equivalent to a CW complex

So we get to the point:

Theorem 3.6.1. *Any metrizable ANR is homotopy equivalent to some absolute CW-complex.*

Let Y be a metrizable ANR. From Theorem (2.6.9) we know that there exists a simplicial polytope K with the Whitehead topology which dominates Y - that is, there exist mappings

$$\Phi : K \to Y$$

$$\Psi : Y \to K$$

such that the composed map $\Phi \circ \Psi : Y \to Y$ is homotopic to Id_Y.

From Example (3.4.3) we know that K is in fact an absolute CW-complex. The following is then clear:

Proposition 3.6.2. *Any metrizable ANR is dominated by an absolute CW-complex.* □

Hence Theorem (3.6.1) follows from the following theorem:

Theorem 3.6.3. *A space which is dominated by an absolute CW-complex is homotopy equivalent to some absolute CW-complex.*

Proof. Assume that the situation is as described above.

The mappings Φ and Ψ induce homomorphisms

$$\Phi_* : \pi_n(K, k_0) \to \pi_n(Y, \Phi(k_0)); \qquad \Psi_* : \pi_n(Y, y_0) \to \pi_n(K, \Psi(y_0))$$

between homotopy groups for all $k_0 \in K$ and $y_0 \in Y$ where $n \in \mathbb{N}$ and functions between families of path components in the case of $n = 0$. Since $\Phi \circ \Psi \simeq Id_Y$ there exists for all $n \in \mathbb{N}_0$ an isomorphism (or bijection in the case $n = 0$) $s_n : \pi_n(Y, \Phi(k_0)) \to \pi_n(Y, y_0)$ such that $s_n \circ (\Phi \circ \Psi)_* = Id_{\pi_n(Y,y_0)}$; hence even $(\Phi \circ \Psi)_*$ is an isomorphism. It follows that Φ_* is a surjection and that Ψ_* is an injection for all $n \in \mathbb{N}_0$.

The mapping Φ_* is not generally an isomorphism. However, we may adjoin cells to K and obtain a larger absolute CW-complex L in such a way that Φ extends to a function Φ' defined on the new space which induces isomorphisms $\Phi'_* : \pi_n(L, l_0) \to \pi_n(Y, \Phi'(l_0))$ for all $l_0 \in L$ and for all $n \in \mathbb{N}$ and a bijection when $n = 0$. We may define a mapping $\Psi' : Y \to L$ by setting $\Psi' = i \circ \Psi$ where $i : K \hookrightarrow L$ is the inclusion mapping. Then if $y \in Y$,

$$(\Phi' \circ \Psi')(y) = \Phi'(\Psi'(y)) = \Phi'(\Psi(y)) = \Phi(\Psi(y)) = (\Phi \circ \Psi)(y)$$

and it follows that $\Phi' \circ \Psi' = \Phi \circ \Psi \simeq Id_Y$. Now there exist again isomorphisms $s_n : \pi_n(Y, \Phi'(k_0)) \to \pi_n(Y, y_0)$ such that $s_n \circ (\Phi' \circ \Psi')_* = Id_{\pi_n(Y,y_0)}$; then $\Phi'_* \circ \Psi'_* = (\Phi' \circ \Psi')_*$

is an isomorphism and since Φ'_* is an isomorphism for all n then so is Ψ'_*. It follows that

$$(\Psi' \circ \Phi')_* : \pi_n(L, l_0) \to \pi_n(L, (\Psi' \circ \Phi')(l_0))$$

is an isomorphism for all $l_0 \in L$ and hence $\Psi' \circ \Phi'$ is a homotopy equivalence.

Now if $f : L \to L$ is a homotopy inverse of $\Psi' \circ \Phi'$, then we obtain

$$\Psi' \circ \Phi' \simeq (\Psi' \circ \Phi') \circ (\Psi' \circ \Phi' \circ f) = \Psi' \circ (\Phi' \circ \Psi') \circ \Phi' \circ f \simeq \Psi' \circ Id_Y \circ \Phi' \circ f = \Psi' \circ \Phi' \circ f \simeq Id_L.$$

We already know that $\Phi' \circ \Psi' \simeq Id_Y$, and hence $\Phi' : L \to Y$ is a homotopy equivalence with homotopy inverse $\Psi' : Y \to L$.

Hence it only remains to construct the absolute CW-complex L with the extended mapping $\Phi' : L \to Y$ such that $\Phi'_* : \pi_n(L, l_0) \to \pi_n(Y, \Phi'(l_0))$ is an isomorphism for all $n \in \mathbb{N}$ and a bijection in the case $n = 0$, for all base points $l_0 \in L$ and such that $\Phi'|K = \Phi$.

We will construct the space L and the mapping Φ' by inductively adding cells and extending the mapping one step at a time.

We set $M^0 = K$ and $L^0 = (M^0)^0 = K^0$. Now M^0 is an absolute CW-complex containing K and L^0 is its 0-skeleton.

First choose some base point $k_0 \in K$ and consider the induced mapping

$$\Phi_* : \pi_0(M^0, k_0) = \pi_0(K, k_0) \to \pi_0(Y, \Phi(k_0))$$

between the families of path components of K and Y, respectively. It is surjective, as stated earlier. If Φ_* is injective then it is a bijection and we may set $L^1 = K$. In case Φ_* is not injective, then we have some mapping $f_i : (S_i^0, 1) = (S^0, 1) \to (K, k_0)$ such that $f_i \simeq \epsilon$ is not true but $\Phi \circ f_i \simeq \epsilon$ (That is, $f_i(-1)$ and $f_i(1)$ lie in different path components of K but are taken to the same path component of Y by Φ).

Since by Theorem (3.4.19) (K, K^0) is 0-connected, every path component of K intersects K^0; hence we may define a homotopy $H^i : S_i^0 \times I \to K$ such that $H_0^i = f_i$ and $H_1^i = g_i$ where $g_i : S_i^0 \to K^0$. Now we may use this mapping g_i to adjoin a 1-cell e_i^1 to $K^0 = L^0$.

Since $[\Phi \circ g_i] = 0$ it follows that $(\Phi \circ g_i)(S_i^0)$ is contained in one path component and so there exists an extension $\gamma_i : \overline{B}_i^1 \to Y$ of $\Phi \circ g_i$.

We repeat this procedure for each $i \in J$, where $\{[f_i] : i \in J\} = \{[f] \in \pi_0(K) | [f] \neq 0 \wedge [\Phi \circ f] = 0\}$, and then repeat the whole thing for each base point $k_0 \in K$.

Let $\{e_i^1 : i \in I\}$ be the set of all 1-cells added and let their attaching maps be f_i for all $i \in I$.

Now define the adjunction space $M^1 = K \cup \bigcup_{i \in I} e_i^1$; this M^1 is an absolute CW-complex (See Example (3.4.4). Also define $L^1 = (M^1)^1$, the 1-skeleton of M^1.

In order to extend the mapping Φ we define a new mapping

$$\Phi^* : K \dot{\cup} \dot{\bigcup}_{i \in I} \overline{B}_i^1 \to Y$$

by setting

$$\Phi^*(k) = \Phi(k) \text{ if } k \in K; \qquad \Phi^*(t) = \gamma_i(t) \text{ if } t \in \overline{B}_i^1$$

Now we may define a function

$$\Phi^1 : M^1 \to Y$$

by setting

$$\Phi^1(m) = \Phi^*(x)$$

where $x \in \pi^{-1}(m)$ where $\pi : K \dot{\cup} \dot{\bigcup}_{i \in I} B_i^1 \to M^1$ is the canonical projection. Now, since π is a quotient map and Φ^* is continuous, Φ^1 is also continuous. Furthermore,

$$\Phi^1_* : \pi_0(M^1, k_0) \to \pi_0(Y, \Phi^1(k_0))$$

is a bijection for each base point $k_0 \in K$. It is then easy to see that this map is in fact a bijection for all basepoints $k_0 \in M^1$.

Now let $n \in \mathbb{N}$ and assume that we have found an absolute CW-complex M^n and a mapping $\Phi^n : M^n \to Y$ such that K is a subcomplex of M^n and such that

$$\Phi^n_* : \pi_k(M^n, m_0) \to \pi_k(Y, \Phi^n(m_0))$$

is a bijection for $k = 0$ and an isomorphism for $k \in \{1, ..., n-1\}$ for all base points $m_0 \in L^n$, where $L^n = (M^n)^n$.

Consider the induced mapping

$$\Phi^n_* : \pi_n(M^n, m_0) \to \pi_n(Y, \Phi^n(m_0)).$$

Since $\Phi^n \circ \Psi^n = \Phi \circ \Psi \simeq Id_Y$ it follows that $\Phi^n_* \circ \Psi^n_*$ is an isomorphism and we see that Φ^n_* is a surjection for all base points m_0. In case Φ^n_* is injective for all m_0, set $M^{n+1} = M^n$ and $\Phi^{n+1} = \Phi^n$. In case Φ^n_* is not injective for each base point m_0, denote by

$$\{[f_i] : i \in I\}$$

the set of generators of $Ker\Phi^n_*$. Then the maps $f_i : (S^n, s_0) \to (M^n, m_0)$ are continuous.

Let $i \in I$. Then $f_i : (S^n, s_0) \to (M^n, L^n)$ where $dimension(S^n - s_0) \leq n$ and (M^n, L^n) is n-connected by Theorem (3.4.18), hence by Lemma (3.4.16) there exists a mapping $g_i : S^n \to L^n$ such that $f_i \simeq g_i$. Now we use this mapping g_i to adjoin an $n+1$-cell e_i^{n+1} to $(M^n)^n$, and we do the same for each $i \in I$. Repeat the whole procedure for all base points $m_0 \in L^n$, and denote by $\{e_i^{n+1} : i \in I'\}$ the set of new cells, and let their attaching maps be f_i for all $i \in I'$.

We define the adjunction space $M^{n+1} = M^n \cup \bigcup_{i \in I} e_i^{n+1}$ and $L^{n+1} = (M^{n+1})^{n+1}$, which makes M^{n+1} an absolute CW-complex containing K and L^{n+1} its $n+1$-skeleton (See Example (3.4.4) for a proof that M^{n+1} is an absolute CW-complex).

Since for each $i \in I$ $[f_i] = [g_i]$ we have that $\Phi^n \circ g_i \simeq \epsilon$ where $\epsilon : S^n \to Y$ is a constant map. We may extend the constant map ϵ to $\overline{B}^{n+1}$, and thus since the inclusion $S^n \to \overline{B}^{n+1}$ is a cofibration, we may extend the whole homotopy to $\overline{B}^{n+1} \times I$. It follows that there exists a continuous extension $(\Phi^n)_i^*$ of $\Phi^n \circ g_i$ to $\overline{B}^{n+1} = \overline{B}_i^{n+1}$. Note that the only reason why it is possible to extend this mapping $\Phi^n \circ g_i$ is that it is homotopic to the constant map!

We define the mapping $(\Phi^{n+1})^* : M^n \dot{\cup} \dot{\bigcup}_{i\in I} \overline{B}_i^{n+1} \to Y$ by setting

$$(\Phi^{n+1})^*(x) = \Phi^n(x) \text{ if } x \in M^n; \qquad (\Phi^{n+1})^*(x) = (\Phi^n)_i^*(x) \text{ if } x \in \overline{B}_i^{n+1}$$

Because $(\Phi^{n+1})^*(x) = (\Phi^n)_i^*(x) = (\Phi^n \circ f_i)(x) = (\Phi^{n+1})^*(f_i(x))$ whenever $x \in S_i^n = \partial \overline{B}_i^{n+1}$ there is a well-defined function $\Phi^{n+1} : M^{n+1} \to Y$ given by

$$\Phi^{n+1}(m) = (\Phi^{n+1})^*(x)$$

where $x \in \pi^{-1}(m)$ and where $\pi : M^n \dot{\cup} \dot{\bigcup}_{i\in I} \overline{B}_i^{n+1} \to M^{n+1}$ is the canonical projection. Because π is a quotient map Φ^{n+1} is continuous.

Claim: Now $(\Phi^{n+1})_* : \pi_k(M^{n+1}, m_0) \to \pi_k(Y, y_0)$ where $y_0 = \Phi^{n+1}(m_0)$ is an isomorphism when $k = 1, ..., n$ and a bijection when $k = 0$ for all base points $m_0 \in L^n$.

Proof: Let $k \in \{1, 2, ..., n-1\}$ and let $g : (S^k, s) \to (M^{n+1}, m_0)$ where $m_0 \in L^n$ be such that $\Phi_*^{n+1}([g]) = 0 \in \pi_n(Y, y_0)$. Then since $dimension(S^k - s) \leq n$ and (M^{n+1}, M^n) is n-connected there exists a mapping $h : S^k \to M^n$ such that $g \simeq h$ rel x_0. Then

$$[\Phi^n \circ h] = [\Phi^{n+1} \circ g] = 0 \in \pi_k(Y, y_0)$$

and since Φ_*^n is an isomorphism, h is nullhomotopic in M^n. But then g is nullhomotopic in M^{n+1} and so Φ_*^{n+1} is injective. It is easy to see that Φ_*^{n+1} is also surjective since Φ_*^n was.

For the homotopy classes $[f_i]$ where $i \in I$ we now have that $f_i \simeq g_i = \pi \circ i$ where $i : S^n \hookrightarrow \overline{B}_i^{n+1} \hookrightarrow M^n \dot{\cup} \dot{\bigcup}_{i\in I} \overline{B}_i^{n+1}$ and since $\overline{B}_i^{n+1}$ is contractible, i is nullhomotopic, and thus so is $\pi \circ i = f_i$. Hence $[f_i] = 0 \in \pi_n(M^{n+1}, m_0)$ for all base points $m_0 \in L^n$.

Since the $[f_i]$ were the generators of $Ker \Phi_*^n$ and all the elements of $\pi_n(M^{n+1}, m_0)$ correspond to elements of $\pi_n(M^n, m_0)$ the $[f_i]$ are also generators of $Ker \Phi_*^{n+1}$. But since they are all zero in $\pi_n(M^{n+1}, m_0)$, it follows that $Ker \Phi_*^{n+1} = 0$. Hence $\Phi_*^{n+1} : \pi_n(M^{n+1}, m_0) \to \pi_n(Y, y_0)$ is injective and it is of course surjective since Φ^n_* was surjective.

Finally, when constructing M^{n+1} from M^n we did not add any new path components; hence $\pi_0(M^{n+1}, m_0) \cong \pi_0(M^n, m_0)$ and so Φ^{n+1}_* is a bijection when $k = 0$ for all base points $m_0 \in L^n$.

Furthermore, one can easily see that the argument holds also for base points $m_0 \in L^{n+1}$. $\square$

We define the CW-complex L to be the one whose n-skeleton is L^n for all $n \in \mathbb{N}_0$ and we define the mapping $\Psi' : L \to Y$ by setting

$$\Psi'(x) = \Psi^n(x) \text{ if } x \in L^n$$

Now the mapping Ψ' is continuous because it is continuous on every L^n.

Claim: The mapping

$$\Psi'_* : \pi_n(L, l_0) \to \pi_n(Y, \Psi'(l_0))$$

is an isomorphism when $n \in \mathbb{N}$ and a bijection when $n = 0$ for all $l_0 \in L$.

Proof: Let $n \in \mathbb{N}$. Now Ψ'_* is surjective since all of the Ψ^n_* were so. Assume that $[f] \in Ker\Psi'_*$. Then $f : (S^n, s) \to (L, l_0)$, but since S^n is compact, then so is $f(S^n)$ and so by Proposition (3.4.6) $f(S^n) \subset L^m$ for some $m \in \mathbb{N}_0$. We may assume $m \geq n$. Now $\Psi^n \circ f = \Psi' \circ f \simeq \epsilon$ rel s where ϵ is the constant map; hence $[f] \in Ker\Psi^n_*$ - but $Ker\Psi^n_* = 0 \in \pi_n(M^m, l_0)$ and hence $[f] = 0 \in \pi_n(L, l_0)$.

Now assume $n = 0$. Since adding n-cells for $n > 1$ did not add or remove any path components to or from those in L^0 we have that $\Psi'_* : \pi_0(L, l_0) \to \pi_0(Y, \Psi'(l_0))$ is a bijection.

Hence Theorem (3.6.3) has been proved.

$\square$

The proof of Theorem (3.6.1) is hereby completed. $\square$

Corollary 3.6.4. *A topological manifold is homotopy equivalent to a CW-complex.*

Proof. By Theorem (2.7.7) a topological manifold is a metrizable ANR, and hence by Theorem (3.6.1) homotopy equivalent to some CW-complex. $\square$

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